An overview of (completeness in) Lorentzian geometry

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These slides can also be found at

https://www.asc.ohio-state.edu/terekcouto.1/texts/OK_slides_october2023.pdf

Riemannian manifolds – a lightning review

Recall that a Riemannian metric on a smooth manifold M is a smooth assignment of positive-definite inner products g_x on each tangent space T_xM, for every $x \in M$. We call (M, g) a Riemannian manifold.

In particular, each individual tangent space (T_xM, g_x) is linearly isometric to $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product in $\mathbb{R}^n$.

On every Riemannian manifold there is a unique covariant derivative operator ∇, called the Levi-Civita connection, mapping any two vector fields X and Y on M to a third one, $\nabla_X Y$.

Relative to a coordinate system $(x^1, \ldots, x^n)$ for M, we may write

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^{k}\partial_k, \qquad \text{for all } i, j = 1, \ldots, n,$$

where the Γ_{ij}^k are called the Christoffel Symbols of ∇ in the given system.

Geodesics and completeness

A curve $\gamma\colon I \to M$ is called a geodesic if it is a solution of the system of differential equations

$$\ddot{x}^k + \sum_{i,j=1}^{n} \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0, \qquad k = 1, \ldots, n,$$

for every coordinate system $(x^1, \ldots, x^n)$ on M.

We also say that γ is complete if its maximal domain of definition is the entire real line $\mathbb{R}$, and that (M, g) is complete if all of its geodesics are complete. The following theorem is well-known:

Theorem

Compact Riemannian manifolds are complete.

... but we are not here to talk about Riemannian manifolds.

Lorentz-Minkowski space

Consider a point-particle in Euclidean space $\mathbb{R}^3$, moving from its initial position to its final position by following a displacement vector $\mathbf{v} = (\Delta x, \Delta y, \Delta z)$.

As the motion of the particle cannot have a speed greater than the speed of light $c = 1$, we have

$$\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2 < 1,$$

which may be reorganized as $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 < 0$.

Definition

The *n*-dimensional Lorentz-Minkowski space is the pair $\mathbb{R}^n_1 = (\mathbb{R}^n, \langle \cdot, \cdot \rangle_1)$, where the symmetric and non-degenerate bilinear form $\langle \cdot, \cdot \rangle_1$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle_1 = x_1 y_1 + \cdots + x_{n-1} y_{n-1} - x_n y_n,$$

for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Lorentz-Minkowski space

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for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Here, non-degenerate means that even though $\langle \cdot, \cdot \rangle_1$ is not positive-definite, it still induces an isomorphism between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. (i.e., $\mathbf{v} \mapsto \langle \mathbf{v}, \cdot \rangle_1$)

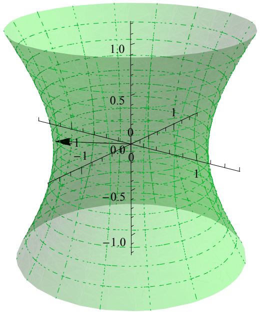
A non-zero vector $\mathbf{v} \in \mathbb{R}^n_1$ is called:

- spacelike, if $\langle \mathbf{v}, \mathbf{v} \rangle_1 > 0$;
- lightlike, if $\langle \mathbf{v}, \mathbf{v} \rangle_1 = 0$;
- timelike, if $\langle \mathbf{v}, \mathbf{v} \rangle_1 < 0$;

The type of $\mathbf{v}$, according to the above sorting, is called its causal character.

Lorentz-Minkowski space

For example, for $\mathbf{v} = (x, y, z) \in \mathbb{R}^3_1$, we have that $\langle \mathbf{v}, \mathbf{v} \rangle_1 = x^2 + y^2 - z^2$.
See below the level sets of $\langle \mathbf{v}, \mathbf{v} \rangle_1 = a$, for $a \in \{-1, 0, 1\}$:



$a = 1$

Lorentz-Minkowski space

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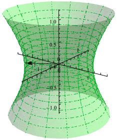

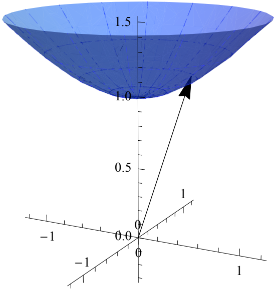

$a = -1$

Lorentz-Minkowski space

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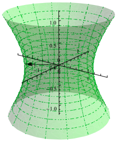

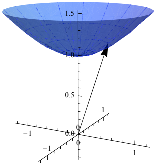

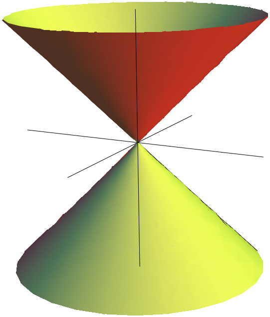

$a = 0$

Lorentz-Minkowski space

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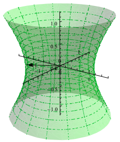

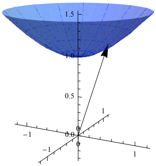

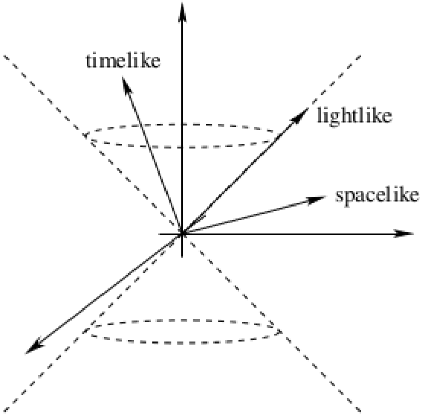

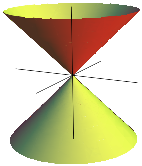

Lorentzian manifolds

Definition

A Lorentzian metric on a smooth manifold M is a smooth assignment of Lorentzian scalar products g_x on each tangent space T_xM, for $x \in M$. We call (M, g) a Lorentzian manifold.

In other words, for a Lorentzian manifold (M, g), each individual tangent space (T_xM, g_x) is linearly isometric to $\mathbb{R}^n_1$ instead of $\mathbb{R}^n$!

The concepts of geodesic and completeness still make sense for Lorentzian manifolds — but completeness becomes much more subtle.

From here on, we'll discuss some of those subtleties and relations between compactness and completeness in the Lorentzian setting.

Multiple notions of Lorentzian completeness

Let (M, g) be a Lorentzian manifold and γ be a geodesic in M. The causal character of γ (spacelike/timelike/lightlike) is the causal character of one of its velocity vectors.

And (M, g) is said to be spacelike-complete (resp. timelike-complete or lightlike-complete) if all of its spacelike (resp. timelike or lightlike) geodesics are complete.

Theorem

The three types of Lorentzian completeness are logically independent.

This conclusion was obtained after a series of examples due to Kundt (1963), Geroch (1968), and Beem (1976).

Multiple notions of Lorentzian completeness

To continue with the discussion, we need the concept of curvature.

If (M, g) is either Riemannian or Lorentzian, its Riemann curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and it is the obstruction for (M, g) to be locally isometric to $\mathbb{R}^n$ or $\mathbb{R}^n_1$.

Theorem (Lafuente López, 1988)

For any locally symmetric Lorentzian manifold (that is, satisfying $\nabla R = 0$), all three types of completeness are equivalent.

Remark: It is an open question whether $\nabla^k R = 0$ with $k \geq 2$ implies that the three types of completeness are equivalent. (Contrast with the Riemannian case, where $\nabla^k R = 0$ for some $k \geq 2$ implies that $\nabla R = 0$.)

Back to compactness!

Homogeneity

Recall that for any Riemannian or Lorentzian manifold (M, g), an isometry of (M, g) is a diffeomorphism $f\colon M \to M$ such that

$$\mathrm{g}_{f(x)}(\mathrm{d}f_x(\mathbf{v}), \mathrm{d}f_x(\mathbf{w})) = \mathrm{g}_x(\mathbf{v}, \mathbf{w}),$$

for all $x \in M$ and $\mathbf{v}, \mathbf{w} \in T_xM$.

There is a natural group action of the group $\mathrm{Iso}(M, \mathrm{g})$ on M, given by evaluation. We call (M, g) homogeneous if such action is transitive.

Theorem (Marsden, 1972)

Every compact homogeneous Lorentzian manifold must be complete.

Remark: This result is true for metrics of more general indefinite metric signature, not just Lorentzian. When $\dim M = 3$, it suffices to assume that (M, g) is locally homogeneous (Dumitrescu-Zeghib, 2010).

Completeness with special vector fields

A vector field $\mathbf{X}$ on a Riemannian or Lorentzian manifold (M, g) is called a Killing vector field if its flow consists of isometries of (M, g). In terms of Lie derivatives, $\mathcal{L}_{\mathbf{X}}\mathrm{g} = 0$.

Theorem (Kamishima, 1993)

Every compact Lorentzian manifold with constant sectional curvature, admitting a timelike Killing field, must be complete.

However, the two assumptions in Kamishima's theorem are too restrictive!

More generally, we say that $\mathbf{X}$ is a conformal Killing vector field if its flow consists of conformal transformations of (M, g). In terms of Lie derivatives, $\mathcal{L}_{\mathbf{X}}\mathrm{g} = f\mathrm{g}$ for some function f.

Theorem (Romero-Sánchez, 1995)

Every compact Lorentzian manifold admitting a timelike conformal Killing vector field must be complete.

Completeness with curvature conditions

Another direction in which Kamishima's theorem was generalized consisted in removing the Killing field assumption instead of the constant curvature condition. A first result in this direction actually came before Kamishima:

Theorem (Carrière, 1989)

Every compact flat Lorentzian manifold must be complete.

Carrière's theorem was then extended:

Theorem (Klingler, 1996)

Every compact Lorentzian manifold with constant sectional curvature must be complete.

Remark: As a consequence of Klingler's theorem, together with a classical result due to Calabi and Markus (1962), it follows that there are no compact Lorentzian manifolds with constant positive sectional curvature.

More recent results

Further conditions implying completeness become much more subtle.

A Lorentzian manifold (M, g) is called a pp-wave spacetime if it admits a parallel lightlike field $\mathbf{X}$ such that $R(\mathbf{X}^\perp, \mathbf{X}^\perp, \cdot, \cdot) = 0$.

Theorem (Leistner-Schliebner, 2016)

Every compact pp-wave spacetime must be complete.

The above result combined with Klingler's result and a little more extra work shows the following:

Corollary (Leistner-Schliebner, 2016)

Every indecomposable and locally symmetric $(\nabla R = 0)$ compact Lorentzian manifold must be complete.

More recent results

As in with Kamishima's theorem, the existence of a parallel lightlike field together with the pp-wave condition may sometimes be too restrictive. But the pp-wave condition may be dropped:

Theorem (Mehidi-Zeghib, 2022)

A compact Lorentzian manifold admitting a parallel lightlike vector field must be complete.

However, dropping the lightlike vector field condition is not possible without paying the price with some other curvature condition:

Theorem (Derdzinski-T., 2023)

A generic compact Lorentzian manifold with parallel Weyl curvature which is not conformally flat or locally symmetric must be complete.

Remark: Here, genericity refers to a technical condition satisfied by "almost all" (M, g) satisfying the other conditions ($\nabla W = 0$, $W \neq 0$, $\nabla R \neq 0$).

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Thank you for your attention!