Killing vector fields on compact pseudo-Kähler $\partial\bar\partial$ manifolds are holomorphic

Andrzej Derdzinski & Ivo Terek†

Department of Mathematics – The Ohio State University

†terekcouto.1@osu.edu

Abstract

It is well-known that Killing vector fields on compact Kähler manifolds are necessarily real-holomorphic. Here, we generalize this result — with two different proofs — to compact pseudo-Kähler manifolds (that is, without assuming that the metric is positive-definite), under the additional assumption that the underlying complex manifold has the $\partial\bar\partial$ property. Lastly, in real dimension 4, the $\partial\bar\partial$ property turns out to be unnecessary. We do not know if the $\partial\bar\partial$ assumption can be removed in higher dimensions.

Introduction/Preliminaries

By a *pseudo-Kähler manifold* we mean a pseudo-Riemannian manifold (M, g) endowed with a ∇-parallel almost-complex structure J, where ∇ is the Levi-Civita connection of g, which consists of isometries $J_x\colon T_xM \to T_xM$, for each $x \in M$. In this setting, J is automatically integrable. In addition, we say that (M, g) is a *pseudo-Kähler $\partial\bar\partial$-manifold* if the underlying complex manifold satisfies the $\partial\bar\partial$ lemma:

every closed ∂-exact of $\bar\partial$-exact (p, q) form equals $\partial\bar\partial\lambda$ for some $(p-1, q-1)$-form λ. $\quad(\partial\bar\partial)$

For example, when M is compact and g is Riemannian, $(\partial\bar\partial)$ holds. Manifolds of Fujiki class $\mathcal{C}$ also satisfy $(\partial\bar\partial)$.

A vector field v on (M, g) is called:

(i) a *Killing vector field* if $\mathcal{L}_v\mathrm{g} = 0$.

(ii) *real-holomorphic* if $\mathcal{L}_vJ = 0$.

If $\omega = \mathrm{g}(J\cdot, \cdot)$ is the Kähler form of (M, g), we note that (ii) above is equivalent to $\mathcal{L}_v\omega = 0$ (in view of $\nabla\mathrm{g} = 0$).

Objective

Investigate the relation between Killing and real-holomorphic vector fields in indefinite metric signature.

Main results

In general dimension

Theorem A. Every Killing vector field on a compact pseudo-Kähler $\partial\bar\partial$-manifold is real-holomorphic.

In any complex manifold M, every closed $(p, 0)$-form is holomorphic; when M is compact and $(\partial\bar\partial)$ holds, the converse holds and every holomorphic differential form is closed, cf. **[2, p. 269]** and **[3, p. 101]**. As a consequence, we have that

whenever $(\partial\bar\partial)$ holds, if a $(p, 0)$-form ζ (with $p \geq 1$) is such that $\partial\zeta$ is closed, then $\partial\zeta = 0$. $\quad(1)$

Now, if v is g-Killing, the 1-form

$$\xi = \mathrm{g}(Jv, \cdot) - i\mathrm{g}(v, \cdot) \qquad (2)$$

is of bidegree $(1, 0)$, with

$$\partial\xi = \mathrm{g}([\mathcal{L}_vJ], \cdot) - i\mathrm{g}([\mathcal{L}_vJ]J\cdot, \cdot), \qquad \bar\partial\xi = i\mathrm{g}((J[\nabla v]J - \nabla v)\cdot, \cdot). \qquad (3)$$

With this in place, $\partial\xi$ is parallel, and hence closed. Therefore $\partial\xi = 0$ due to (1). Taking the real part, we see that $\mathcal{L}_vJ = 0$, as desired.

An alternative proof of Theorem A

Whenever a compact complex manifold has the $\partial\bar\partial$ property, there is a Hodge decomposition

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}M, \qquad (4)$$

where each $H^{p,q}M$ consists of Dolbeaut cohomology classes of closed (p, q)-forms. A complex cohomology class is *real* if it contains a real-valued closed differential form, and this happens if and only if, for every p, q, the $H^{q,p}$-component equals the conjugate of the $H^{p,q}$-component.

Lemma. For any ∇-parallel real 2-form α on a complex manifold M with a Kähler connection (i.e., $\nabla J = 0$), such that $\alpha(J\cdot, J\cdot) = -\alpha$, the complex-valued 2-form $\alpha - i\alpha(J\cdot, \cdot)$ is holomorphic. If, in addition, M is compact and $(\partial\bar\partial)$ holds, then α being exact implies that $\alpha = 0$.

Whenever v is g-Killing, the Lie derivative $\mathcal{L}_v\omega$ is parallel and exact (by Cartan's homotopy formula). As

$$[\mathcal{L}_vJ]J = -J[\mathcal{L}_vJ] \quad \text{and} \quad \mathcal{L}_v\omega = \mathrm{g}([\mathcal{L}_vJ]\cdot, \cdot), \qquad (5)$$

the lemma above applies to $\alpha = \mathcal{L}_v\omega$.

The real-dimension four case

Theorem B. In real-dimension four, the conclusion of Theorem A holds without the $\partial\bar\partial$ property.

Here, we start by noting that the vector bundle endomorphisms $C\colon TM \to TM$ which are, at every point $x \in M$, g_x-skew-adjoint (that is, $C^* = -C$), form the sections of the vector subbundle $\mathfrak{so}(TM, \mathrm{g})$ of $\mathrm{End}_{\mathbb{R}}(TM)$.

We let $\mathcal{E}$ denote the vector subbundle of $\mathfrak{so}(TM, \mathrm{g})$ consisting of the endomorphisms which are, in addition, $\mathbb{C}$-antilinear: $C^* = -C$ and $JC = -CJ$. It holds that

$\mathcal{E}$ is a complex vector bundle of rank $m(m-1)/2$, where $m = \dim_{\mathbb{C}} M$, with a pseudo-Hermitian fiber metric whose real part is g. $\quad(6)$

Whenever v is g-Killing,

the Lie derivative $\mathcal{L}_vJ$ is a parallel section of $\mathcal{E}$, such that $(\mathcal{L}_vJ, \mathcal{L}_vJ)_{L^2} = 0$, $\quad(7)$

as a consequence of a general fact: any exact p-form on a compact pseudo-Riemannian manifold is L^2-orthogonal to all parallel p-times covariant tensor fields.

When $m = 2$, $\mathcal{E}$ is a *line* bundle due to (6), and its pseudo-Hermitian fiber metric must be positive or negative definite. The same must now hold for its g-induced real part. Hence, (7) implies that $\mathcal{L}_vJ = 0$, as required.

Conclusion

- Killing vector fields on compact pseudo-Kähler $\partial\bar\partial$ manifolds are real-holomorphic.
- The $\partial\bar\partial$ condition can be dropped in real-dimension four.
- It is still an open question whether the $\partial\bar\partial$ assumption may be dropped in higher dimensions.

Acknowledgements

We would like to thank Kirollos Masood for bringing Yamada's 2012 paper to our attention, Fangyang Zheng for very useful comments regarding the $\partial\bar\partial$ condition, and Takumi Yamada for a brief but helpful communication.

References

[1] A. Derdzinski and I. Terek, *Killing vector fields in compact pseudo-Kähler manifolds*. To appear in J. Geom. Anal.

[2] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), pp. 245–274.

[3] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*. Lecture Notes in Math. **439**, Springer-Verlag, 1975.

[4] T. Yamada, *Holomorphic vector fields of compact pseudo-Kähler manifolds*. J. Geom. Phys. **62** (2012), pp. 740–750.