Codazzi tensors in homogeneous spaces

(joint work with James Marshall Reber)

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These slides can also be found at

https://www.asc.ohio-state.edu/terekcouto.1/texts/GSTGC_slides_april2024.pdf

The covariant exterior derivative

The exterior derivative on a smooth manifold M is a collection of operators $\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)$, characterized by the Palais formula:

$$\mathrm{d}\omega(X_0,\ldots,X_k) = \sum_{i=0}^{k}(-1)^{i-1}X_i(\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_k)) + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots\widehat{X}_j,\ldots,X_k).$$

Whenever E is a vector bundle over M, we may consider E-valued forms and set $\Omega^k(M;E) = \Omega^k(M)\otimes_{C^\infty(M)}\Gamma(E)$. Given a linear connection $\nabla\colon \mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$, we may define the covariant exterior derivative $\mathrm{d}^\nabla\colon \Omega^k(M;E)\to\Omega^{k+1}(M;E)$ by

$$\mathrm{d}^\nabla\omega(X_0,\ldots,X_k) = \sum_{i=0}^{k}(-1)^{i-1}\nabla_{X_i}(\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_k)) + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots\widehat{X}_j,\ldots,X_k).$$

The covariant exterior derivative

When $E = M \times \mathbb{R}$ is the trivial line bundle over M and ∇ is the standard flat connection on E, then $\Omega^k(M;E) = \Omega^k(M)$ and $\mathrm{d}^\nabla = \mathrm{d}$.

But in general, it is no longer true that $(\mathrm{d}^\nabla)^2 = 0$.

In fact, for $\psi \in \Omega^0(M;E) = \Gamma(E)$, we have that:

$$[(\mathrm{d}^\nabla)^2\psi](X,Y) = R^\nabla(X,Y)\psi$$

The curvature is in fact enough to describe other higher powers of d^∇:

$$[(\mathrm{d}^\nabla)^3\psi](X,Y,Z) = R^\nabla(X,Y)\nabla_Z\psi + R^\nabla(Y,Z)\nabla_X\psi + R^\nabla(Z,X)\nabla_Y\psi$$

Even higher:

$$\begin{aligned}[(\mathrm{d}^\nabla)^4\psi](X,Y,Z,W) &= \{R^\nabla(X,Y), R^\nabla(Z,W)\}\psi \\ &\quad + \{R^\nabla(Z,X), R^\nabla(Y,W)\}\psi + \{R^\nabla(X,W), R^\nabla(Y,Z)\}\psi.\end{aligned}$$

Codazzi tensor fields and examples

Let (M, g) be a Riemannian manifold and ∇ be its Levi-Civita connection.

Definition

A Codazzi tensor field on (M, g) is a twice-covariant symmetric tensor field A on M with $\mathrm{d}^\nabla A = 0$, when A is regarded as a T^*M-valued 1-form.

Explicitly, A is Codazzi if and only if

$$(\nabla_X A)(Y, Z) = (\nabla_Y A)(X, Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$ or, equivalently, if ∇A is fully symmetric.

Example (1)

If A is parallel, then A is Codazzi. In particular, if $A = \lambda \mathrm{g}$ for some $\lambda \in \mathbb{R}$.

Example (2)

(M, g) has harmonic curvature if and only if Ric is a Codazzi tensor field, in view of $\mathrm{div}\, R = \mathrm{d}^\nabla \mathrm{Ric}$.

Codazzi tensor fields and examples

Definition: A is Codazzi $\iff$ $(\nabla_X A)(Y,Z) = (\nabla_Y A)(X,Z)$

Example (3)

(M,g) has harmonic Weyl curvature if and only if Sch is a Codazzi tensor field, in view of $\operatorname{div} W = \mathrm{d}^\nabla \mathrm{Sch}$.

Example (4)

If (M,g) has constant sectional curvature K, and $f \in \mathrm{C}^\infty(M)$ is arbitrary, then $A_f = \operatorname{Hess} f + Kf\mathrm{g}$ is Codazzi. (Uses $\mathrm{d}^\nabla \operatorname{Hess} f = R(\cdot,\cdot,\nabla f,\cdot)$.)

Every Codazzi tensor field on (M,g) is locally of the form A_f for some f.

Example (5)

If (M^3,g) is a gradient-type Ricci soliton — that is, $\mathrm{Ric} + \operatorname{Hess} f = \lambda\mathrm{g}$ for some $f \in \mathrm{C}^\infty(M)$ and $\lambda \in \mathbb{R}$ — then $A = \mathrm{e}^{-f}(\mathrm{Ric} - \mathrm{scal}\,\mathrm{g}/2)$ is Codazzi.

Codazzi tensor fields and examples

Example (6)

If $(\overline{M}, \overline{\mathrm{g}})$ is a Riemannian manifold and M is a two-sided hypersurface carrying a unit normal field ξ and second fundamental form II — so that $\overline{\mathrm{g}}(S(X), Y) = \mathrm{II}(X, Y)\xi$ for the shape operator S of M — then

$$[\overline{R}(X, Y)Z]^{\perp} = [(\mathrm{d}^{\nabla}\mathrm{II})(X, Y)Z]\xi \qquad \text{(Ricci equation)}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

If $(\overline{M}, \overline{\mathrm{g}})$ has constant sectional curvature K, then

$$\overline{R}(X, Y)Z = K(\overline{\mathrm{g}}(Y, Z)X - \overline{\mathrm{g}}(X, Z)Y) \qquad (\overline{R} = K\bar{\mathrm{g}} \owedge \bar{\mathrm{g}})$$

is tangent to M.

Hence II is Codazzi.

Some of what is known

Here are some well-known results on Codazzi tensors in (M, g):

- Codazzi operators — obtained from Codazzi tensors via g — commute with the Ricci operator (Bourguignon '81).
- For a Codazzi tensor with constant eigenvalues, all eigendistributions are integrable and have totally geodesic leaves. (Widely well-known.)
- On the open set consisting of points admitting neighborhoods on which the eigenvalue functions are smooth and with constant multiplicities, all eigendistributions are integrable and have totally umbilic leaves (Derdzinski, '80).
- If a Codazzi tensor has $\dim M$ mutually distinct eigenvalues, then all Pontryagin forms of M vanish (Derdzinski-Shen, '83).
- If a Lie group equipped a left-invariant Riemannian metric has a non-parallel Codazzi tensor field, then it has both strictly positive and strictly negative sectional curvatures (d'Atri, '85).

Reductive homogeneous spaces

A Riemannian manifold (M, g) is homogeneous if the natural action of $\mathrm{Iso}(M, \mathrm{g})$ on M is transitive. Homogeneous spaces can always be expressed in the form G/H, where G is a Lie group and H is a closed subgroup of G.

Definition

A homogeneous space G/H is reductive if there exists a vector-space direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with $\mathfrak{m}$ being $\mathrm{Ad}(H)$-invariant.

The projection $\pi \colon G \to G/H$ induces an isomorphism $\mathrm{d}\pi_e \colon \mathfrak{m} \cong T_{eH}(G/H)$.

This means that in the same way the geometry of a connected Lie group G is controlled by its Lie algebra $\mathfrak{g}$, the geometry of a reductive homogeneous space G/H is controlled through $\mathfrak{m}$.

Projecting the Lie bracket in $\mathfrak{g}$ onto $\mathfrak{m}$, we obtain a nonassociative algebra $(\mathfrak{m}, [\cdot, \cdot]_{\mathfrak{m}})$.

Correspondences for reductive homogeneous spaces

Fact: the G-equivariant sections of a G-equivariant smooth fiber bundle $E \to G/H$ are in one-to-one correspondence with elements of the fiber E_{eH} which are fixed by H. (If $\phi \in E_{eH}$ is fixed, set $\psi_{gH} = g \cdot \phi$.)

Examples:

- G-invariant vector fields on G/H are in one-to-one correspondence with elements in $\mathfrak{m}$ which are fixed by $\mathrm{Ad}(H)$.
- G-invariant tensor fields on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-invariant tensors (of the same type) on $\mathfrak{m}$.
- G-invariant distributions on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-invariant vector subspaces of $\mathfrak{m}$. A G-invariant distribution $\mathcal{P}$ on G/H is integrable if and only if $\mathcal{P}_{eH}$ is closed under $[\cdot,\cdot]_{\mathfrak{m}}$ (Tondeur, '65).

Lastly, G-invariant connections ∇ on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-equivariant multiplications $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ (Nomizu, '54).

What happens in $\mathfrak{m}$

Putting all of this together, we see that a twice-covariant G-invariant tensor field A on G/H is Codazzi if and only if

$$\alpha(X, A)(Y, Z) = \alpha(Y, A)(X, Z)$$

holds in $\mathfrak{m}$.

($\alpha(X, A)(Y, Z) \doteq -A(\alpha(X, Y), Z) - A(Y, \alpha(X, Z))$ corresponds to ∇A.)

Diagonalizing A, we may write

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r,$$

where $\mathfrak{m}_i$ is the eigenspace associated with λ_i, and we order $\lambda_1 < \cdots < \lambda_r$.

Definition

A subalgebra $\mathfrak{k}$ of $\mathfrak{m}$ is called totally geodesic if $\mathfrak{k}$ is closed under α, that is, $\alpha(X, Y) \in \mathfrak{k}$ whenever $X, Y \in \mathfrak{k}$.

The algebraic structure of Codazzi tensors in G/H

Theorem (Marshall Reber–T., '23)

Whenever A is a G-invariant Codazzi tensor field on G/H, there is an eigenspace decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ into mutually orthogonal $\mathrm{Ad}(H)$-invariant totally geodesic subalgebras of $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$, and the compatibility condition

$$(\lambda_i - \lambda_k)^2 \big\langle [X_i, Y_j]_{\mathfrak{m}}, Z_k \big\rangle + (\lambda_j - \lambda_i)^2 \big\langle [X_i, Z_k]_{\mathfrak{m}}, Y_j \big\rangle = 0 \qquad (\dagger)$$

holds for all $X, Y, Z \in \mathfrak{m}$ and $i, j, k \in \{1, \ldots, r\}$.

Conversely, if a direct sum decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ into mutually orthogonal $\mathrm{Ad}(H)$-invariant vector subspaces is given and $(\dagger)$ holds, any choice of mutually distinct real constants $\lambda_1, \ldots, \lambda_r$ gives rise to a G-invariant Codazzi tensor field on G/H via $A = \bigoplus_{i=1}^r \lambda_i \langle \cdot, \cdot \rangle|_{\mathfrak{m}_i \times \mathfrak{m}_i}$.

In addition, $\nabla A \neq 0$ if and only if there are mutually distinct indices i, j, k with $\langle X_i, [Y_j, Z_k]_{\mathfrak{m}} \rangle \neq 0$, in which case A has ≥ 3 distinct eigenvalues.

The canonical connection

The canonical connection of second kind on G/H is the connection ∇^0 corresponding to the zero product in $\mathfrak{m}$. Its curvature tensor is given by $R^0(X,Y)Z = -[[X,Y]_{\mathfrak{h}}, Z]$.

In view of the Jacobi identity

$$\sum_{\text{cyc}}[[X,Y]_{\mathfrak{h}}, Z] + \sum_{\text{cyc}}[[X,Y]_{\mathfrak{m}}, Z] = 0,$$

we see that $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}})$ is a Lie algebra if and only if ∇^0 satisfies the first Bianchi identity.

We can also define the sectional curvature K^0 by

$$K^0(\mathbb{R}X \oplus \mathbb{R}Y) = \frac{\langle R^0(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}.$$

To state our next result, we consider the difference curvature $K^d = K - K^0$.

$\exists$ Codazzi $\implies$ mixed-sign curvatures

Theorem (Marshall Reber–T., '23)

If G/H has a G-invariant Codazzi tensor field A with $\nabla A \neq 0$, the difference sectional curvature K^d assumes both positive and negative values.

Briefly, the proof consists in carefully analyzing the expression

$$K^d(\Pi) = \frac{2}{(\lambda_i - \lambda_j)^2} \sum_k (\lambda_i - \lambda_j)(\lambda_j - \lambda_k) \|[X_i, Y_j]_k\|^2,$$

for $\Pi = \mathbb{R}X_i \oplus \mathbb{R}Y_j$, and choosing suitable indices i and j.

Example

When G/H is naturally reductive (i.e., $\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle = 0$), every Codazzi tensor is parallel. Indeed, in this case one has

$$K^d(\Pi) = \frac{1}{4}\|[X, Y]_{\mathfrak{m}}\|^2 \geq 0$$

whenever $\{X, Y\}$ is an orthonormal basis of Π.

The bigger picture

Recall: $\operatorname{div} R = \mathrm{d}^{\nabla}\mathrm{Ric}$ holds for any Riemannian manifold (M, g).

Back to harmonic curvature: $\operatorname{div} R = 0$.

Conjecture (Aberaouze–Boucetta, '22)

Any homogeneous Riemannian manifold with harmonic curvature must have parallel Ricci tensor.

Here are some instances on where the conjecture is true:

- In dimension 4 (Podesta–Spiro, '95; Haji-Badali–Zaeim, '15)
- When M is $\mathbb{S}^n$ or $\mathbb{RP}^n$ (Peng–Qian, '16)
- When M is in a certain class of compact Lie groups (Wu–Sun, '22)
- When M is naturally reductive (Marshall Reber–T., '23).

Some references

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Thank you for your attention!

(scan here for more
on my research)