# MATH2177 – AU23 – RECITATION DIARY

This is a diary for the **MATH2177 - Mathematical Topics for Engineers** recitation classes offered at OSU, on the Autumn 2023 term. No additional material or exercises will be added here, as it is meant to be a reasonably faithful reflection of what happens in class (although I cannot promise I won't add a remark or footnote here or there, elaborating further on things I particularly find interesting). Since it is unlikely that all three recitations will be 100% equal all the time, you may occasionally see an exercise or explanation that was not discussed in class (but this only means that it may have been discussed in one of the other sections). Most of the exercises and problems discussed will be taken from the course textbook [1].

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# 1 August 22nd

## 1.1 The Big Picture$^{TM}$

This class is divided into four major parts:

(1) Multivariable Calculus;

(2) Linear Algebra;

(3) Ordinary Differential Equations;

(4) Fourier Series.

There's no need to justify Multivariable Calculus here (after all, this is a natural continuation of MATH1172). Linear Algebra is too useful of a subject for us to spend the entire semester without using it (although most of you may have to take an actual Linear Algebra class later). Ordinary Differential Equations: briefly speaking, those are equations where one solves for a function instead of a number, and there are derivatives involved in the given equation (the word "ordinary" refers to the fact that only functions of a single variable are involved; "partial differential equations" are those with partial derivatives involved). Fourier Series: you can vaguely think of those as a trigonometric analogue of the Taylor series you have seen before:

$$f(x) \sim \sum_{n\geq 0} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{versus} \quad f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Just like the coefficients of a Taylor series are given in terms of derivatives of $f$, there are specific formulas for the coefficients $a_n$ and $b_n$ of a Fourier series, but we don't need to worry about that now. The pressing question, however, is how these four topics make sense together.

To answer it, consider the **Heat Flow Problem**: imagine you have a metal wire of length $L$, positioned in the $x$-axis of a cartesian plane, with endpoints at $x = 0$ and $x = L$. Assume that:

(i) we are given the initial temperature distribution $f(x)$ of the wire (i.e., a point of coordinate $x$ in the wire has temperature equal to $f(x)$; in particular, non-constant $f$ means that the temperature distribution along the wire is not uniform);

(ii) we are given the diffusivity coefficient $\beta$ of the wire;

(iii) the temperature at the endpoints of the wire will always be the same throughout time (say, 0 in some appropriate scale).

**Question:** Can we predict future temperature distributions along the wire? (i.e., can we find a function $u(x,t)$ giving us the temperature at the point of coordinate $x$ at the wire $t$ minutes after the initial measurement?)

This sort of question is the basis of the mathematical modeling for problems of physical nature: find a function controlling the time evolution of the physical system you're interested in studying. Such a function is usually a solution of some partial differential equation.

**Answer:** Yes! It boils down to solving the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \beta \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L. \end{cases} \tag{H}$$

It is, *prima facie*, a very complicated problem. Think back of the four major parts of this course (1)-(4) mentioned above. The time-dependent temperature distribution $u(x,t)$ is a function of more then one variable (1), dealing with the boundary conditions $u(0,t) = u(L,t) = 0$ becomes easier with the use of Linear Algebra (2), which in turn allows us to reduce the partial differential equation to an ordinary differential equation (3), and the final answer to (H) can then be conveniently expressed in terms of the Fourier Series (4) of the initial temperature distribution $f(x)$.

> **The ultimate goal of this class is to get you to combine all the tools listed above to solve problems such as the Heat Flow Problem (H).**

## 1.2 Critical points and the 2nd derivative test – *redux*

One of the main problems in single-variable calculus is the one of *optimization*: given a function, find all of its critical points and decide on their *local nature* (that is, whether they are a local maximum, local minimum, or inflection point), or to find the global maximum and minimum values of the function on a given set. With this in mind, let us build up new concepts from old, relying on your previous knowledge from single-variable calculus.

| Concept | One variable | Two variables |
|---|---|---|
| 1st derivative | The number $f'(x)$ | The vector $\nabla f(x,y) = \big(\partial f/\partial x, \partial f/\partial y\big)$ |
| Critical points | $f'(a) = 0$ or DNE | $\nabla f(a,b) = (0,0)$ or DNE |
| 2nd derivative | The number $f''(x)$ | The Hessian matrix $H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ |

**Example 1** (The number of critical points cannot be predicted)

- *Functions may not have critical points at all!* Consider $f(x,y,z) = g(x,y) + z$, where $g(x,y)$ is the most horrible expression you can come up with. Then we have that $\nabla f = (*,*,1)$. As the last component of $\nabla f$ is never zero, then $\nabla f$ can never be the zero vector, and so $f$ has no critical points.
- *Functions may have infinitely many critical points!* Consider $f(x,y) = y^2$, so that $\nabla f(x,y) = (0,2y)$. *All* points of the form $(a,0)$ are critical points of $f$, that is, all points in the $x$-axis are critical.

When doing multivariable calculus, **we need to abandon the prime notation for derivatives**. For example, $f'(x,y)$ does not tell you whether you're supposed to compute the first derivative with respect to $x$ or $y$. It is an ambiguous notation which will cause confusion. This means that we have no choice but to use the Leibniz (fraction) notation for partial derivatives. But it may get cumbersome to write, and for this reason the following shorthand (subscript notation) for partial derivatives became standard:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}.$$

This is the notation used in the table when describing $H(x,y)$.

The philosophies behind the gradient $\nabla f(x,y)$ and the Hessian[1] $H(x,y)$ are the same: the gradient is a mathematical device to gather all information about the first-order derivatives of a function into a single object (namely, a vector), while the Hessian does the same for the second-order derivatives (there are four of them now).

There are many changes from the single-variable case to the two-variables case, but increasing the number of variables even further makes little-to-no conceptual difference: for a function of 1000 variables, the gradient vector has 1000 entries and the Hessian matrix has size $1000 \times 1000$. For this reason we will focus mostly on functions of two variables (and sometimes of three).

To decide on the local nature of critical points, the main tool we had in single-variable calculus was the second derivative test, looking at the sign of $f''(a)$ at a critical point $a$. But what should be the analogue of this? What does it mean for the matrix $H(x,y)$ to be positive or negative?

**Definition 1**

The **determinant** of $H$ is the function $D$ given by $D = f_{xx}f_{yy} - f_{xy}^2$.

For now, you may think of $D$ as **the obstruction for applying the single-variable second derivative test with $f_{xx}$ playing the role of the old $f''$**. If at a critical point $(a,b)$, we have $D(a,b) > 0$, then the single-variable second derivative test may be

[1] By the way, here's a quick way of computing $H(x,y)$: writing $\nabla f = (f_x, f_y)$, the first row of $H$ is the gradient of the first component function $f_x$, while the second row of $H$ is the gradient of the second component function $f_y$. In other words, just keep computing gradients!

used with $f_{xx}$! It also works with $f_{yy}$ instead of $f_{xx}$, but let's always do it with $f_{xx}$ for the sake of consistency.

| One variable | Two variables |
|---|---|
| If $f''(a) > 0$, then $a$ is a local minimum ($\ddot\smile$) | If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $(a,b)$ is a local minimum ($\ddot\smile$) |
| If $f''(a) < 0$, then $a$ is a local maximum ($\ddot\frown$) | If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $(a,b)$ is a local maximum ($\ddot\frown$) |
| If $f''(a) = 0$, the test fails | If $D(a,b) = 0$, the test fails |

**Remark.**

(i) A very common mistake is to think that $f''(a) = 0$ means that $a$ is a saddle (or inflection point, these words are synonymous). This is not true. Case in point: for the function $f(x) = x^4$ and $a = 0$ ($f(0) = f'(0) = f''(0) = f'''(0) = 0$ but $f^{(4)}(0) > 0$ and $a$ is a global minimum). If $f''(a) = 0$ but $f'''(a) \neq 0$, then $a$ is a saddle point. If $f'''(a) = 0$, then we must look at the fourth derivative $f^{(4)}(a)$: if positive, then $a$ is a local minimum, and if negative, $a$ is a local maximum. If $f^{(4)}(a) = 0$, we must look at the fifth derivative. If $f^{(5)}(a) \neq 0$, then $a$ is a saddle point. If $f^{(5)}(a) = 0$, we must look at the sixth derivative $f^{(6)}(a)$: if positive, then $a$ is a local minimum, and if negative, $a$ is a local maximum. This procedure repeats, alternating between even-order derivatives and odd-order derivatives, until a conclusion is obtained. The "proof" of this fact relies on the Taylor polynomial of $f$. There is no easy analogue of this trick for functions of more than one variable.

(ii) There is a version of the second derivative test for functions with more than two variables. The number of conditions to be considered increases with the number of variables. Justifying it requires more Linear Algebra than what we have available now. See Sylvester's Criterion (for positivity of matrices) for more details.

As item (i) in the above remark suggests, there is no general way of deciding whether a critical point $a$ with $f''(a) = 0$ is a saddle/inflection point without looking at even higher order derivatives. But in the **two-variables case**[2], we have an extra conclusion:

$$\textbf{if } D(a,b) < 0\textbf{, then } (a,b) \textbf{ is a saddle point}.$$

Note: if $D(a,b) < 0$, the conclusion is immediate and you don't need to look at the sign of $f_{xx}$ or any other second-order derivatives – if $D(a,b) < 0$, it's game over!

[2] This is very specific to two variables!

**Example 2**

*Find and classify the critical points of the function* $f(x,y) = (3x-2)^2 + (y-4)^2$.

We may start by computing the gradient of $f$ with the aid of the chain rule, as

$$\nabla f(x,y) = \big(2\cdot(3x-2)^1\cdot 3\ ,\ 2\cdot(y-4)^1\cdot 1\big) = \big(18x-12, 2y-8\big).$$

This equals the zero vector $(0,0)$ if and only if $(x,y) = (2/3,4)$, and so this is the only critical point of $f$. To decide its nature, we apply the second derivative test: the Hessian matrix of $f$ is given by

$$H(x,y) = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix},$$

but observe that, in general, **this will depend on** $x$ **and** $y$. In this example, $H(x,y)$ is a constant matrix because the given function $f$ is simple enough, and there is nothing deeper about it. Evaluating $H(x,y)$ at all critical points is a separate step which must be done carefully. As

$$H(2/3,4) = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix},$$

we see that $D(2/3,4) = 36 > 0$, and thus we can apply the single-variable second derivative test with $f_{xx}$ playing the role of $f''$. As $f_{xx}(2/3,4) = 2 > 0$, we have that $(2/3,4)$ is a local minimum.

Let us do a quick sanity-check. Letting $u = 3x-2$ and $v = y-4$, we may abuse notation and write $f(u,v) = u^2+v^2$. The graph of this function in the $uvz$-space is called a **paraboloid**, and it is obtained by rotating the classical parabola around the $z$-axis. This means that the graph of $f$ considered in the original $xyz$-space is a paraboloid, up to this change of variables (which amounts to an offset and a stretching of the $x$-axis). Geometrically, we see that the critical point is described by $u = v = 0$, which means that $x = 2/3$ and $y = 4$. Thus the only critical point $(2/3,4)$ is in fact a global minimum for $f$. Changing variables like this will be useful not only to try and gain geometric intuition for situations like this (where what you have in front of you is similar to something you have already seen before, but not quite equal to it), but it will also be a very important tool when dealing with change of variables for computing double integrals.

Note, however, that sometimes it is not so simple to get the answer so quickly by using geometric intuition. This is why we have techniques such as the multivariable second derivative test: we can gain information about the function even without knowing how to draw its graph.

**Example 3**

*Find and classify the critical points of the function* $f(x,y) = x^2 + xy^2 - 2x + 1$.

To find the critical points of $f$, we must compute the gradient of $f$ and set it equal to the zero vector $(0,0)$. In this case, we have that the gradient equals $\nabla f(x,y) = (2x + y^2 - 2, 2xy)$, and we must consider the system

$$\begin{cases} 2x + y^2 - 2 = 0, \\ 2xy = 0. \end{cases}$$

Recall here the **key idea: whenever the product of two factors equals zero, one of them must necessarily be zero**. So, the second equation says that either $x = 0$ or $y = 0$. But we have no way to tell which one, and so we must consider both cases.

- Case 1: If $x = 0$, then $y^2 - 2 = 0$ and so $y = \pm\sqrt{2}$. This gives us the pair of critical points $(0, \sqrt{2})$ and $(0, -\sqrt{2})$.
- Case 2: If $y = 0$, then $2x - 2 = 0$ and so $x = 1$. This gives us the single critical point $(1,0)$.

It remains to classify these critical points. To do so, we compute the Hessian of $f$ as

$$H(x,y) = \begin{bmatrix} 2 & 2y \\ 2y & 2x \end{bmatrix},$$

as well as its determinant $D(x,y) = 4x - 4y^2$. Then:

- $(1,0)$: here, we have that $D(1,0) = 4 > 0$, so we can think of the single-variable second derivative test with $f_{xx}$ playing the role of $f''$. Since we have that $f_{xx}(1,0) = 2 > 0$, we conclude that $(1,0)$ is a local minimum of $f$.
- $(0, \sqrt{2})$: this time, we have $D(0, \sqrt{2}) = -8 < 0$, so $(0, \sqrt{2})$ is a saddle point.
- $(0, -\sqrt{2})$: this time, we have $D(0, -\sqrt{2}) = -8 < 0$, so $(0, -\sqrt{2})$ is a saddle point.

The fact that the critical points $(0, \pm\sqrt{2})$ had the same local nature should not be a surprise: the relation $f(x,y) = f(x,-y)$ says that $f$ is "even in the variable $y$" (which geometrically says that the graph of $f$ is symmetric about the $xz$-plane in space).

# 2 August 29th

## 2.1 Lagrange multipliers

We continue with the discussion on maxima and minima of functions of two variables. Most of the time, the region $R$ to be considered will be closed and bounded (those are called "compact", in short), and it may be thought of as having two regions: an **interior**, and a **boundary**.

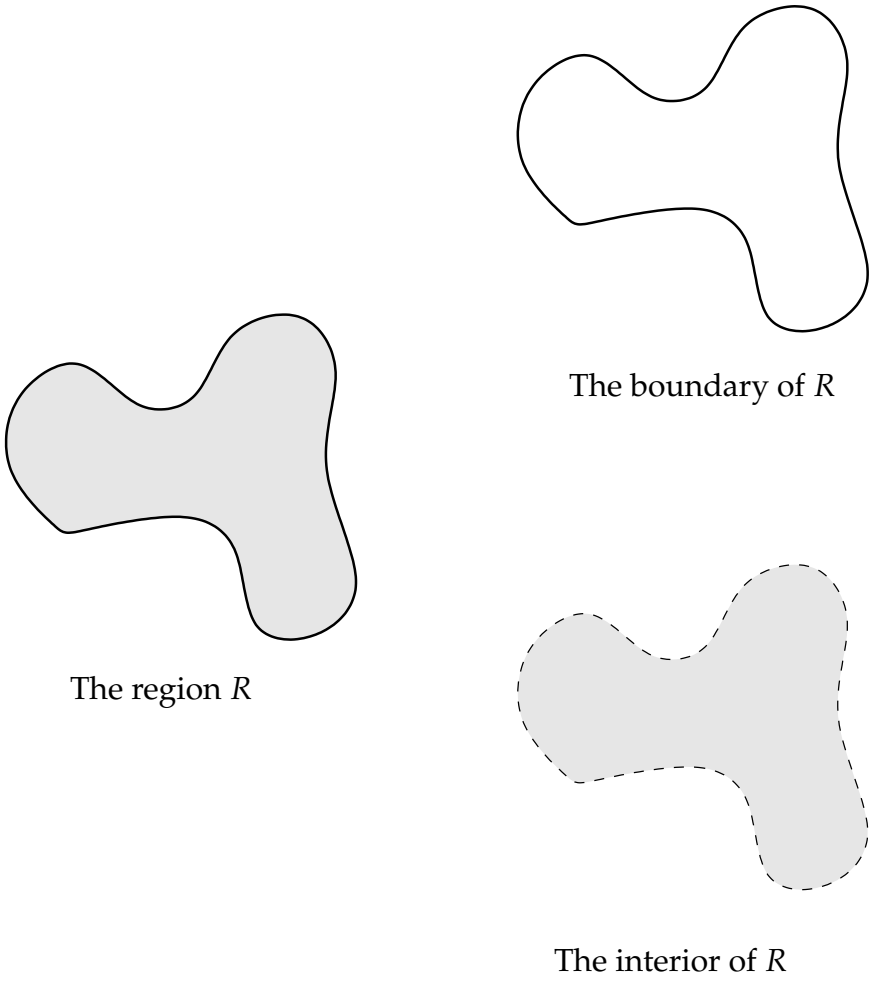



Figure 1: Describing the interior and boundary of a planar region $R$.

The problem here is:

> **How to find the global maximum and minimum values of a function $f$ of two variables in such a planar region $R$?**

The idea to approach this is: we find all **candidates** to points achieving such maximum and minimum values, list all of them in a table, evaluate the function $f$ of all such points, and compare the resulting values.

Now, there are two types of candidates:

(i) Candidates on the interior of $R$: those are just the critical points of $f$ which happen to land in the interior of $R$.

(ii) Candidates on the boundary of $R$: here, the situation becomes more subtle, and there are two ways to proceed.

- Parametrize (each component of) the boundary of $R$ and reduce everything to a single-variable calculus problem.
- Apply Lagrange multipliers.

**Remark.** Observe that the second derivative test is only good for determining the local behavior of critical points, **not the global one**. This means that if you're asked to find the global maximum and minimum values of a function, you should not waste time applying the second derivative test to the critical points found in the interior of $R$.

The really new technique here is that of **Lagrange multipliers**. This technique, to be explained below, is only good for when the boundary of $R$ can be described by a **single** equation, in the form $g(x,y) = 0$.

**Theorem 1** (Lagrange multipliers, two variables)

The global maximum and minimum values of a function $f$ of two variables on a set of the form $\{(x,y)\colon g(x,y) = 0\}$ satisfy, provided that $\nabla g(x,y) \neq (0,0)$ whenever $g(x,y) = 0$, the relation

$$\nabla f(x,y) = \lambda \nabla g(x,y),$$

for some real number $\lambda$.

**Remark.** This works when $f$ and $g$ are both functions of three, four, or a billion variables. For more general versions of this procedure, you would have to take a Multivariable Analysis class. The condition $\nabla g(x,y) \neq (0,0)$ whenever $g(x,y) = 0$ is a technical condition that ensure that the curve described by $g(x,y) = 0$ is "nice" (e.g., without corners). All examples we will see in practice will satisfy this assumption, **so do not worry too much about it**. Of course, $f$ and $g$ are assumed to be differentiable functions and with continuous partial derivatives.

**Example 4**

*Find the global maximum and global minimum values of the function*

$$f(x,y) = x^2 + y^2 - 2y + 1$$

*on the region* $R = \{(x,y) \mid x^2 + y^2 \leq 4\}$.

We'll always organize our work in two parts. Observe that $R$ describes a closed disk with center in $(0,0)$ and radius 2. It is always a good idea to try and draw the region $R$, to gain some geometric intuition about the problem. Observe how the interior is described with the strict inequality, and the boundary with the equality.

See Figure 2 ahead on page 12. We proceed:

(a) Candidates on the interior of $R$: the interior consists of the points $(x, y)$ satisfying the relation $x^2 + y^2 < 4$. By replacing the inequality $\leq$ with the strict inequality $<$, we are forgetting about the boundary circle and considering only the open disk bounded by it. We compute the gradient of $f$ as $\nabla f(x, y) = (2x, 2y - 2)$. Setting this equal to $(0, 0)$, we obtain $(x, y) = (0, 1)$. **Is the point $(0, 1)$ in the interior of $R$?** Yes. The reason why we ask ourselves this is because $f$ is defined everywhere (in particular, its domain is larger than $R$), so a priori it could be that critical points found here lie outside $R$. If this were to happen, such critical points would have to be ignored.

(b) Candidates on the boundary of $R$: we use Lagrange multipliers, letting $g(x, y) = x^2 + y^2 - 4$. In this case, $\nabla g(x, y) = (2x, 2y)$ cannot be the zero vector whenever we have $x^2 + y^2 = 4$, which means that the assumptions for using Lagrange multipliers are satisfied. Thus, we have that

$$\begin{cases} \nabla f(x, y) = \lambda g(x, y) \\ g(x, y) = 0 \end{cases} \implies \begin{cases} 2x = 2\lambda x \\ 2y - 2 = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$

Note how the equation $g(x, y) = 0$ is crucial here: without it, we would have three unknowns ($x$, $y$, and $\lambda$), but only two equations, and this would be a big problem. In addition, we do not want to risk finding a "fake candidate" that actually landed outside of the boundary of $R$.

Consider the first equation $2x = 2\lambda x$. We would like to cancel $x$ on both sides, but this step cannot be made if $x = 0$. This means that we need to see what happens when $x = 0$ separately. Thus, we have cases to analyze.

- Case 1: if $x \neq 0$. Then $\lambda = 1$, and substituting this onto the second equation gives $2y - 2 = 2y$, and thus $-2 = 0$. This is clearly nonsense, which says that Case 1 does not happen, and so we get no candidates here.
- Case 2: if $x = 0$. In this case, we only have to solve for $y$, and then the third equation reads $0^2 + y^2 = 4$, so that $y = 2$ or $y = -2$. This case gave us two candidates, $(0, 2)$ and $(0, -2)$.

(c) Candidates on the boundary of $R$ without using Lagrange multipliers: consider the parametrization $\mathbf{r}(t) = (2\cos t, 2\sin t)$, defined for all $t$ (in fact, $0 \leq t \leq 2\pi$ covers the circle). Then $f(\mathbf{r}(t)) = 5 - 4\sin t$. The maximum value of this expression is 9, for $t = 3\pi/2$, and we have $\mathbf{r}(3\pi/2) = (0, -2)$. The minimum along the curve is 1, for $t = \pi/2$, with $\mathbf{r}(\pi/2) = (0, 2)$ (as we'll see, is not the global minimum of $f$ because the critical point in $\text{int}(R)$ gives a lower value). We have reobtained the candidates $(0, 2)$ and $(0, -2)$.

Conclusion:

| Candidates | Values of $f$ |
|---|---|
| $(0,1)$ | 0 |
| $(0,2)$ | 1 |
| $(0,-2)$ | 9 |

Hence the global maximum is 9, realized at $(0,-2)$, while the global minimum is 0, realized at $(0,1)$.

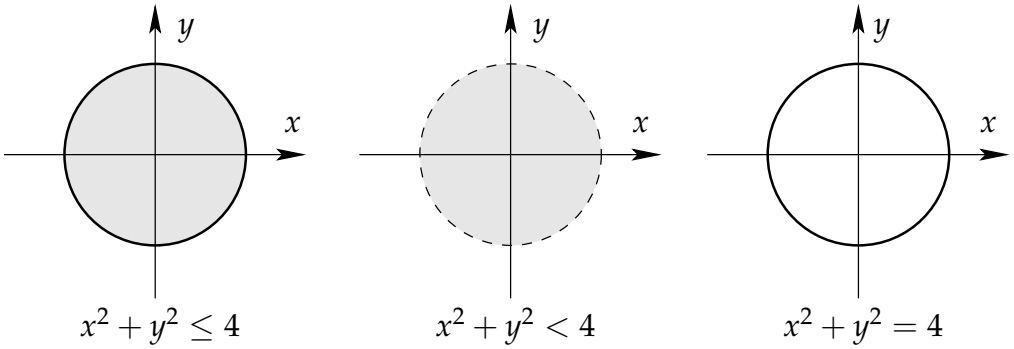



Figure 2: Describing the region $R$ on Example 4 above.

**Example 5**

*Find the global maximum and global minimum values of the function*

$$f(x,y) = 2x^2 - 4x + 3y^2 + 2$$

*on the region* $R = \{(x,y) \mid (x-1)^2 + y^2 \leq 1\}$.

Note that $R$ is a closed disk with center at $(1,0)$ and radius 2.

(a) Candidates on the interior of $R$: the interior consists of the points $(x,y)$ satisfying the relation $(x-1)^2+y^2 < 1$. The gradient of $f$ is $\nabla f(x,y) = (4x-4, 6y)$. Setting this equal to $(0,0)$, we obtain $(x,y) = (1,0)$. **Is the point** $(1,0)$ **in the interior of** $R$**?** Yes, it is actually the center of the disk. Alternatively, one can verify it with algebra: $(1-1)^2 + 0^2 = 0 < 1$.

(b) Candidates on the boundary of $R$: we'll use Lagrange multipliers again, this time letting $g(x,y) = (x-1)^2 + y^2 - 1$. In this case, $\nabla g(x,y) = (2(x-1), 2y)$ cannot be the zero vector whenever we have $(x-1)^2 + y^2 = 1$, which means

that the assumptions for using Lagrange multipliers are satisfied. Thus, we have that

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \implies \begin{cases} 4x - 4 = 2\lambda(x-1) \\ 6y = 2\lambda y \\ (x-1)^2 + y^2 = 1 \end{cases}$$

Consider the second equation $6y = 2\lambda y$. We would like to cancel $y$ on both sides, but this step cannot be made if $y = 0$. Thus, we have cases to analyze.

- Case 1: if $y \neq 0$. Then $\lambda = 3$, and substituting this onto the first equation gives $4x - 4 = 6(x-1)$, from which we obtain $x = 1$. With this, the third equation reads $(1-1)^2 + y^2 = 1$, so $y = 1$ or $y = -1$. This case gave us two candidates $(1,1)$ and $(1,-1)$.
- Case 2: if $y = 0$. In this case, we only have to solve for $x$, and then the third equation reads $(x-1)^2 + 0^2 = 1$, so that $x - 1 = \pm 1$ and hence $x = 0$ or $x = 2$. This case gave us two candidates, $(2,0)$ and $(-2,0)$.

Let's organize everything we have obtained so far in a table again:

| Candidates | Values |
|---|---|
| $(1,0)$ | 0 |
| $(2,0)$ | 2 |
| $(0,0)$ | 2 |
| $(1,1)$ | 3 |
| $(1,-1)$ | 3 |

Hence, the global maximum is 3, realized at $(1,1)$ and $(1,-1)$, while the global minimum is 0, realized at $(1,0)$. It is ok that the global maximum was reached in more than one point (think of the extreme example where $f$ is constant).

We conclude with a guided exercise. It illustrates that Lagrange multipliers cannot be applied when the boundary of the region $R$ cannot be described with a single nice function (the telltale sign is when the boundary of $R$ has sharp corners).

**Exercise 1**

*Find the global maximum and global minimum values of the function*

$$f(x,y) = x^2 + y^2 - 2x - 2y$$

*on the region $R$ bounded by the triangle of vertices at $(0,0)$, $(2,0)$ and $(0,2)$.*

The boundary of $R$ has three sides:

$$\begin{aligned} A &= \{(0,y) \mid 0 \leq y \leq 2\} \\ B &= \{(x,0) \mid 0 \leq x \leq 2\} \\ C &= \{(x,y) \mid y = -x+2 \text{ and } 0 \leq x \leq 2\} \end{aligned}$$

We proceed with our analysis as before:

(a) Candidates on the interior of $R$: this time we have $\nabla f(x,y) = (2x-2, 2y-2)$, so setting this equal to $(0,0)$ gives $(x,y) = (1,1)$. **Is the point $(1,1)$ in the interior of $R$?** No. In fact, $(1,1)$ lies on side $C$, and so it is not considered a candidate **as far as the interior of $R$ is concerned**.

(b) Candidates on the boundary of $R$: we'll study each of the sides $A$, $B$ and $C$ separately.

- Side $A$: evaluating $f$ alongside Side $A$, we are led to consider the composition $A(t) = f(0,t) = t^2 - 2t$, defined on the interval $[0,2]$. The candidates here will be the points corresponding to the endpoints of the interval $[0,2]$, and critical points of $A$ inside the open interval $]0,2[$. The graph of $A$ is a parabola which is concave up, and so $A'(t) = 2t - 2$ leads to $t = 1$. Side $A$ thus gives us the candidates $(0,0)$, $(0,2)$ and $(0,1)$.
- Side $B$: can you draw conclusions from the work done for Side $A$ given that the function $f$ satisfies the symmetry $f(x,y) = f(y,x)$?
- Side $C$: what is the line equation for Side $C$? Consider the composition

$$C(t) = f(t, \text{"line equation"}(t)),$$

defined on $[0,2]$. As before the endpoints of $[0,2]$ gives us the candidates $(2,0)$ and $(0,2)$ (both repeated). Now find which candidates are coming from $C'(t) = 0$, repeating what was done for side $A$.

Summarizing it:

| Candidates | Values of $f$ |
|---|---|
| $(0,0)$ | ? |
| $(2,0)$ | ? |
| $(0,2)$ | ? |
| $(0,1)$ | ? |
| from Side B | ? |
| from Side C | ? |

Fill the table and read off it the desired conclusions.

# 3 September 5th

## 3.1 Starting with double integrals

Today we start with double integrals. Recall the geometric intuition: if $f > 0$, then $\iint_R f(x,y)\,\mathrm{d}A$ computes the volume bounded between the $xy$-plane and the graph of $f$, in the same fashion that $\int_a^b f(x)\,\mathrm{d}x$ computed the area under the graph of $f$. The main difficulty here, however, is to deal with bounds of integration when setting up iterated integrals.

**Example 6**

*Compute* $\displaystyle\iint_R y\cos(xy)\,\mathrm{d}A$, *where* $R = \{(x,y) \mid 0 \leq x \leq 1$ *and* $0 \leq y \leq \pi/3\}$.

Here, $\mathrm{d}A$ stands for the **infinitesimal area element**, which in rectangular coordinates is just given by $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$. The region $R$ is simply a rectangle, as the bounds for $x$ and $y$ are all constants. Do not be mislead to think that $R$ is a sector of a circle just because there's $\pi$ there: $R$ is not being described in polar coordinates, there's no $\theta$ anywhere. One could set up the iterated integrals as

$$\int_0^{\pi/3}\int_0^1 y\cos(xy)\,\mathrm{d}x\,\mathrm{d}y \quad \text{or} \quad \int_0^1\int_0^{\pi/3} y\cos(xy)\,\mathrm{d}y\,\mathrm{d}x.$$

**Fubini's Theorem** says that the order you choose does not matter, you will obtain the same result regardless of the choice made. Now, it could very well happen that one choice of order leads to a much easier computation than the other. In this case, the second option would require an unpleasant integration by parts, while the first one requires a simple $u$-substitution. Making $u = xy$, so $\mathrm{d}u = y\,\mathrm{d}x$ (as $y$ is a constant from the perspective of $x$), we have that

$$\int y\cos(xy)\,\mathrm{d}y = \int \cos u\,\mathrm{d}u = \sin u = \sin(xy).$$

We don't bother with the constant of integration here because we're dealing with definite integrals, so it would dissapear anyway. Thus

$$\begin{aligned}
\int_0^{\pi/3}\int_0^1 y\cos(xy)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\pi/3} \sin(xy)\Big|_{x=0}^{x=1}\mathrm{d}x \\
&= \int_0^{\pi/3} \sin y\,\mathrm{d}x \\
&= -\cos y\Big|_{y=0}^{y=\pi/3} \\
&= -\frac{1}{2} + 1 \\
&= \frac{1}{2}.
\end{aligned}$$

Next, let's look at more general regions which are not rectangles.

**Example 7**

*Set up iterated integrals for a generic continuous function $f(x,y)$ over the region $R$ given in the picture.*

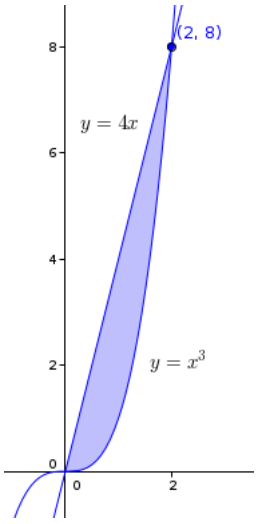
As usual, there are two orders we can set up.

- $\mathrm{d}x\,\mathrm{d}y$. Fix one value of the outermost variable, $y$. What is the range for the other variable $x$, as the horizontal line passing through $y$ cuts the given region, from left to right? Here, we must express $x$ as a function of $y$, so there's some small work to be done. The lower bound for $x$ is $x = y/4$, coming from the line equation, and the upper bound is $x = \sqrt[3]{y}$, from the cubic equation. The variable $y$, in turn, goes from 0 to 8. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^8 \int_{y/4}^{\sqrt[3]{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

- $\mathrm{d}y\,\mathrm{d}x$. Fix one value of the outermost variable, $x$. What is the range for the other variable $y$, as the vertical line passing through $x$ cuts the given region, upwards? Here, we must express $y$ as a function of $x$, so there no work to be done. The lower bound for $y$ is $x^3$, coming from the cubic equation, and the upper bound is $4x$, from the line equation. The variable $x$, in turn, goes from 0 to 2. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^2 \int_{x^3}^{4x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$



Here's another one:

**Example 8**

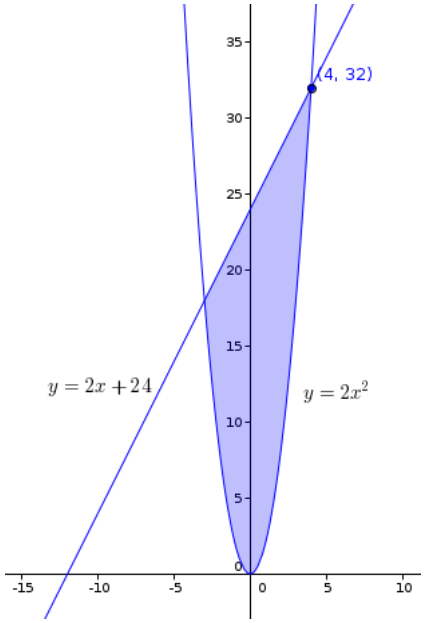
*Set up iterated integrals for a generic continuous function $f(x,y)$ over the region $R$ given in the picture.*

One more time, there are two orders we can set up. The exercise did not give us the coordinates for the left intersection point between the graphs, but we need it to know the full bounds for $x$. To find it, we consider $2x^2 = 2x + 24$, which is readily simplified to $x^2 - x - 12 = 0$. We already know that one of the solutions is $x = 4$. Due to the coefficient 12, the other one is $-3$ or 3, but it clearly cannot be the latter. Hence, the coordinates of the remaining intersection point are $(-3, 18)$ (where 18 is obtained by plugging $x = -3$ into either $y = 2x^2$ or $y = 2x + 24$). Now, let's study what happens with both orders of integration:

- $\mathrm{d}y\,\mathrm{d}x$: Fix one value of the outermost variable, $x$. What is the range for the other variable $y$, as the vertical line passing through $x$ cuts the given region, upwards? Here, we must express $y$ as a function of $x$, so there no work to be done. The lower bound for $y$ is $2x^2$, coming from the quadratic equation, and the upper bound is $2x + 24$, from the line equation. The variable $x$, in turn, goes from $-3$ to 4. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_{-3}^{4}\int_{2x^2}^{2x+24} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

- $\mathrm{d}x\,\mathrm{d}y$: This time, we're forced to break the region into two pieces, as once a value for the outermost variable $y$ is fixed, the lower bound for $x$ cannot be written as a single formula as a function of $y$, due to the "break" at the point $(-3, 18)$. We know that if $R = R_1 \cup R_2$ with $R_1 \cap R_2 = \varnothing$, then the double integral of $f$ over $R$ equals the sum[a] of the double integrals over $R_1$ and $R_2$. Let's say that $R_1$ is the part of $R$ which lies inside the strip $0 \leq y \leq 18$, and $R_2$ is the one inside the strip $18 < y \leq 32$.




Then we have that

$$\begin{aligned}\iint_R f(x,y)\,\mathrm{d}A &= \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_2} f(x,y)\,\mathrm{d}A \\ &= \int_0^{18}\int_{-\sqrt{y/2}}^{\sqrt{y/2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{18}^{32}\int_{(y-24)/2}^{\sqrt{y/2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Namely, the "upper bound" for $x$, once $y$ is fixed, is always $\sqrt{y/2}$, but the lower bound depends on whether $0 \leq y \leq 18$ or $18 < y \leq 32$: in the former case, it is $-\sqrt{y/2}$, and in the latter case it is $(y-24)/2$ (obtained from solving for $x$ in terms of $y$ in $y = 2x + 24$).

[a] This is a two-variable version of the general rule $\int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t$ for single-variable integrals.

By now, you should be convinced that a convenient choice of order of integration is crucial to making things simpler (getting the feeling for which choice is best takes some practice and experience). There are situations, however, where one choice simply makes the problem impossible, and we're forced to switch the order.

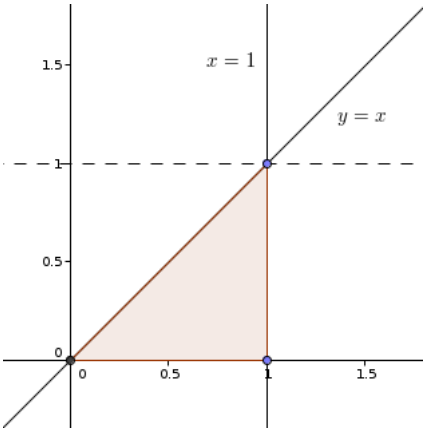
**Example 9**

*Compute* $\displaystyle\int_0^1\int_y^1 \mathrm{e}^{x^2}\,\mathrm{d}x\,\mathrm{d}y$.

The function $f(x) = \mathrm{e}^{x^2}$ has no elementary anti-derivative, in the sense that its indefinite integral cannot be expressed in terms of well-known functions (such as polynomials, rational functions, exponentials, logarithms, and trigonometric functions). Knowing whether a given function of a single-variable has an elemen-

tary anti-derivative or not is not a simple task (keywords: Risch's Algorithm, and Differential Galois Theory). We will not concern ourselves with this. The extra tool we have in this case, is precisely to change the order of integration.

To draw the region of integration, one general strategy is: first recognize that the outermost bounds for $y$ are 0 and 1, so whatever we draw will be inside the region where $0 \leq y \leq 1$. As for the innermost bounds, draw the curves described by the bounds, $x = y$ and $x = 1$. Namely, they're the usual diagonal, and a vertical line.




If the innermost original bounds were from 0 to $y$, the region of integration would be the upper triangle, as opposed to the lower one (as the picture indicates). Now:

$$\begin{aligned}\int_0^1 \int_y^1 e^{x^2}\,dx\,dy &= \int_0^1 \int_0^x e^{x^2}\,dy\,dx \\ &= \int_0^1 x e^{x^2}\,dx \\ &= \frac{e^{x^2}}{2}\bigg|_0^1 = \frac{e-1}{2}.\end{aligned}$$

The $x$ factor produced by realizing the integral with respect to $y$ first saves the day.

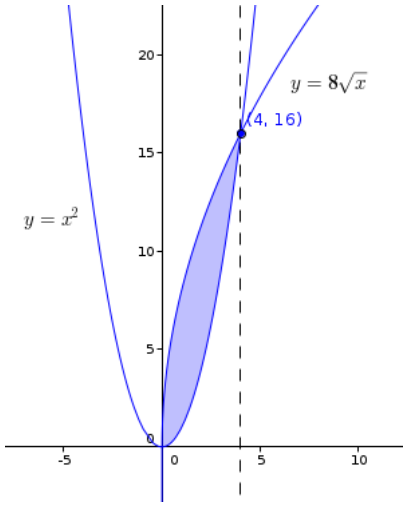
Being able to sketch regions given algebraically is an important skill. Here's more practice:

**Example 10**

*Sketch* $R = \{(x,y) \mid 0 \leq x \leq 4$ *and* $x^2 \leq y \leq 8\sqrt{x}\}$ *and set the iterated integral of a generic continuous function* $f(x,y)$ *over* $R$ *in the order* $dy\,dx$.

We immediately know that whatever we draw will remain inside the vertical strip $0 \leq x \leq 4$. As for $x^2 \leq y \leq 8\sqrt{x}$, forget for one moment that we're dealing with inequalities, and draw the bounds $y = x^2$ and $y = 8\sqrt{x}$ instead. Recall that the graph of $\sqrt{x}$ is obtained by reflecting the graph of $x^2$ about the diagonal line

$y = x$, and that 8 is just a vertical stretching factor (made by design to make $(4, 16)$ the rightmost intersection of the two curves).



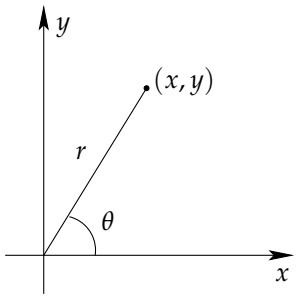

Fixed $x$, the lower bound for $y$ is $x^2$ and the upper bound is $8\sqrt{x}$. As $x$ itself ranges from 0 to 4, we simply have that

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_0^4 \int_{x^2}^{8\sqrt{x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

# 4 September 12th

## 4.1 Polar coordinates

A quick review on polar coordinates: the idea is to represent a point in the plane not by the two numbers $x$ and $y$, but by different numbers. They will be the distance $r$ to the origin, and the angle $\theta$ formed by the $x$-axis and the position vector of $(x, y)$:




To relate the coordinates $r$ and $\theta$ with $x$ and $y$, consider the right triangle with hypothenuse $r$, opposite side to $\theta$ equal to $y$, and adjacent side to $\theta$ equal to $x$, as the figure suggests. Then

$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r}$$

leads to $x = r\cos\theta$ and $y = r\sin\theta$. Finally, the Pythagorean theorem reads $x^2 + y^2 = r^2$.

Next, as far as integration is concerned, we express the infinitesimal area element as $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. When dealing with more general "curvilinear" coordinates $u$ and $v$, we will always have a correction factor, as in $\mathrm{d}A = ?\,\mathrm{d}u\,\mathrm{d}v$. We will understand the general mechanism for this next week. For now, you can take $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$ as a formula we may simply apply when computing double integrals using polar coordinates. Note:

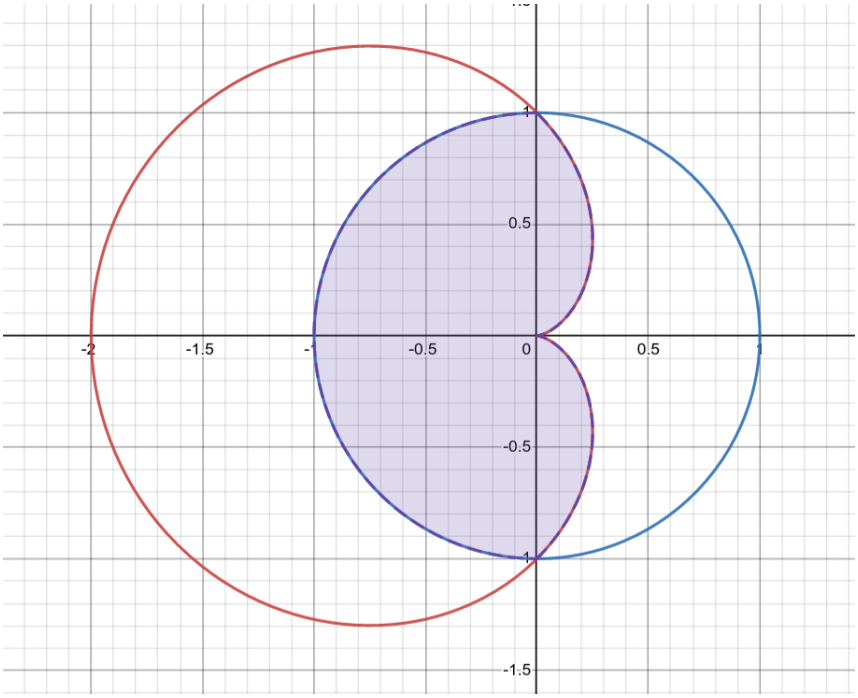
> It is absolutely crucial to not forget $r$ in $r\,\mathrm{d}r\,\mathrm{d}\theta$! This is probably the most common mistake students make in multivariable calculus!

Remembering these small details, you should be able to solve essentially every problem involving polar coordinates. It will also be good to understand what a polar graph is: we express a relation between $r$ and $\theta$ of the form $r = r(\theta)$. For every ray forming angle $\theta$ with the $x$-axis, we mark a point at distance $r(\theta)$ from the origin. Sampling some values of $\theta$ lets us understand which type of curve we're tracing. The next concrete example should help you understand the process:

**Example 11**

*Find the area bounded between the unit circle $r = 1$ and the cardioid $r = 1 - \cos\theta$.*

The "cardio" in "cardioid" hints at the origin of the word and that the curve described by $r = 1 - \cos\theta$ should resemble a heart. See the following picture:




For $0 \leq \theta < \pi/2$, we have that $0 \leq r < 1$, and $r = 1$ for $\theta = \pi/2$ (this indicates the intersection point $(0,1)$). For $\pi/2 < \theta < 3\pi/2$, we have that $r > 1$, so the cardioid doesn't intersect the circle (e.g., for $\theta = \pi$ we see that the point $(-2,0)$ is in the cardioid). For $\theta = 3\pi/2$, we again have $r = 1$ (this indicates the second intersection point $(0,-1)$). The "polar graph" changes at $\theta = \pi/2$ and $\theta = 3\pi/2$, so we should expect that more than one integral is needed here. The region considered is symmetric about the $x$-axis, so we may write its area as

$$A = 2\left(\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta + \int_{\pi/2}^{\pi}\int_0^{1} r\,\mathrm{d}r\,\mathrm{d}\theta\right)$$

Don't forget the correction factor of $r$ in $r\,\mathrm{d}r\,\mathrm{d}\theta$. Now think of "slices" again:

- For a generic $\theta$ between 0 and $\pi/2$, send forward a "radial slice". The lowest value of $r$ in this slice is 0 and the highest is $1-\cos\theta$ (it hits the cardioid before the circle).
- For a generic $\theta$ between $\pi/2$ and $\pi$, send forward a "radial slice". The lowest value of $r$ in this slice is 0 and the highest is 1 (it hits the circle before the cardioid).

Observe that when computing the area $A$, at some point

$$\int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta$$

appears, because of the first integral. To deal with it, we must resort to trigonometric identities. The one that solves the problem here is $\cos^2\theta = (1+\cos(2\theta))/2$.

It is a good idea to review some of these identities. As for the second integral, observe that

$$\int_{\pi/2}^{\pi}\int_{0}^{1} r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{4}$$

for **purely geometrical reasons**: this double integral computes the area of one quadrant of a unit circle, which is simply $\pi/4$! No need to compute that double integral explicitly!

**Remark.** Compare the simple equation $r = 1 - \cos(\theta)$ with its corresponding cartesian equation

$$\sqrt{x^2+y^2} = 1 - \frac{x}{\sqrt{x^2+y^2}}.$$

Employing polar coordinates allows us to express curves with potentially complicated cartesian equations in a simple manner. Another interesting type of curves that can be expressed with polar graphs are *flowers with n petals*: they are given by the relation $r = \cos(n\theta)$, and the angle between two adjacent petals is $2\pi/n$. For example, here's what this looks like when $n = 3$:

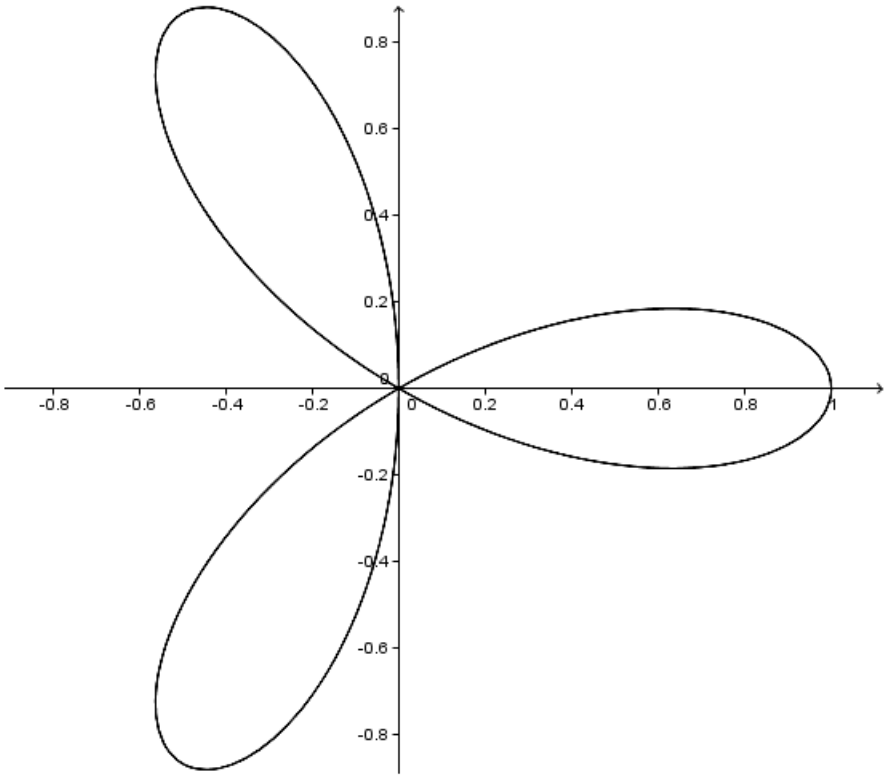


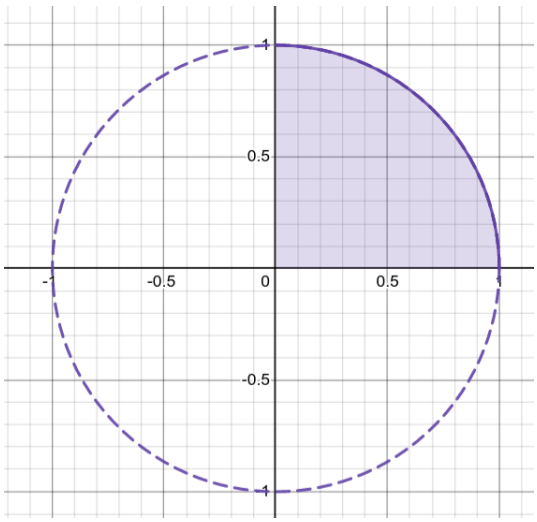

Figure 3: A flower with $n = 3$ petals.

Here's another example, unrelated to polar graphs:

**Example 12**

*Compute* $\displaystyle\iint_R \frac{\mathrm{d}A}{\sqrt{16-x^2-y^2}}$, *where* $R = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4, x \geq 0, y \geq 0\}$.

First of all, we sketch the region of integration:




With this, we move on to the change to polar coordinates:

$$\begin{aligned}
\iint_R \frac{\mathrm{d}A}{\sqrt{16-x^2-y^2}} &= \int_0^{\pi/2}\int_0^2 \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{16-r^2}} \\
&= \left(\int_0^{\pi/2}\mathrm{d}\theta\right)\left(\int_0^2 \frac{r}{\sqrt{16-r^2}}\,\mathrm{d}r\right) \\
&\overset{(*)}{=} \frac{\pi}{2}\left(-\sqrt{16-r^2}\right)\Big|_0^2 \\
&= \frac{\pi}{2}(4-2\sqrt{3}) \\
&= 2\pi - \pi\sqrt{3}.
\end{aligned}$$

On $(*)$, we could have done an $u$-substitution to flesh out more details, as in $u = 16-r^2$, so $\mathrm{d}u = -2r\,\mathrm{d}r$, and thus

$$\int \frac{r}{\sqrt{16-r^2}}\,\mathrm{d}r = \int \frac{1}{\sqrt{u}}\left(-\frac{\mathrm{d}u}{2}\right) = -\frac{1}{2}\int u^{-1/2}\,\mathrm{d}u = -u^{1/2} = -\sqrt{16-r^2}.$$

Note that by doing this $u$-substitution on the side, **we don't risk writing the integrand as a function of $u$ but keeping the bounds for the old variable $r$; this is NOT ok even if you know you'll come back to the initial variable later.**

Next, a fun example (which you should see at least once in your life):

**Example 13**

We know from single-variable calculus that the integral

$$\int e^{-x^2}\,dx$$

cannot be solved, in the sense that there is no elementary anti-derivative for $e^{-x^2}$. So, let's take the impossible and make it worse. Consider

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$

Indulging the lack of self-love of yours truly, let's not stop here and square it:

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2.$$

Now, the fun begins.

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\\ &= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy\end{aligned}$$

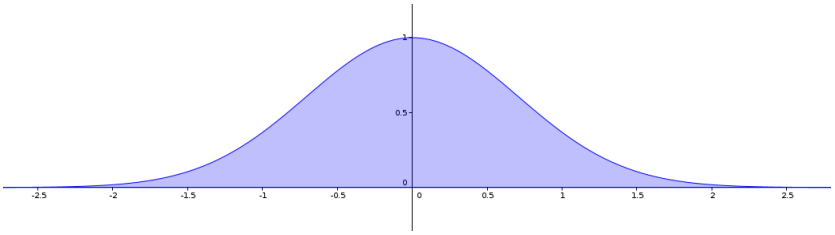
Performing a change of variables to polar coordinates, we continue:

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 &= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2} r\,dr\,d\theta\\ &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{+\infty} re^{-r^2}\,dr\right)\\ &= 2\pi\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^{+\infty}\\ &= \pi(0-(-1))\\ &= \pi.\end{aligned}$$

Note how the correction factor $r$ saved the day. We conclude (as the original definite integral is positive to begin with) that

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

This integral appears in statistics, when studying random variables $X \sim \mathcal{N}(\mu, \sigma^2)$ with normal probability distribution (here, $\mu$ is the mean and $\sigma^2$ is the variance); the graph of $f(x) = \mathrm{e}^{-x^2}$ is the "bell curve" you might be already familiar with:




We have effectively shown that the area under the bell curve equals $\sqrt{\pi}$.

## 4.2 A triple integral

Again: when passing from one variable to two variables, lots of different things happen. But when passing from two variables to a thousand? Everything stays morally the same. Here is one example:

**Example 14**

*Use a triple integral to compute the volume bounded between the plane $z = 0$ and the graph of $z = \sin y$, over the region $D = \{(x, y) : x \leq y \leq \pi \text{ and } 0 \leq x \leq \pi\}$.*

Let's set up the iterated integral in the order $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ (which is one of the most useful when we have a solid bounded by graphs above and below):

$$\iiint_D 1\,\mathrm{d}V = \int_?^? \int_?^? \int_?^? 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Recall that the when computing a volume with a triple integral, the function to be integrated is 1 *by default* (note the general mechanism: to find a volume one integrates $1\,\mathrm{d}V$, for an area one integrates $1\,\mathrm{d}A$, and for arclengths one integrates $1\,\mathrm{d}s$ – more on this last one in a week or two). Sketching the region yourself (or looking at what the book gives you), you can see that the lowest value of $x$ that occurs in $D$ is 0 and the highest is $\pi$. So we have

$$\int_0^\pi \int_?^? \int_?^? 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Fixed a generic value of $x$, what are the bounds for $y$ in terms of $x$? This leads us to

$$\int_0^\pi \int_x^\pi \int_?^? 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Fixed generic values of $x$ and $y$, what are the bounds for $z$ in terms of $x$ and $y$? We

obtain that

$$\int_0^{\pi}\int_x^{\pi}\int_0^{\sin y} 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

is the integral to be computed. Time for a sanity-check: if you were to compute the volume under the graph of $\sin y$ with a double integral, you would have

$$\int_0^{\pi}\int_x^{\pi} \sin y\,\mathrm{d}y\,\mathrm{d}x.$$

Now observe that $\displaystyle\int_0^{\sin y} \mathrm{d}z = \sin y$. This general mechanism explains why we integrate "top minus bottom" when computing the volume of a solid bounded between two graphs, it's like

$$\iint_R \int_{\text{bottom}}^{\text{top}} 1\mathrm{d}z\,\mathrm{d}A = \iint_R (\text{top} - \text{bottom})\mathrm{d}A.$$

# 5 September 19th

## 5.1 Cylindrical coordinates

**Idea:** We have Cartesian coordinates $(x, y, z)$, but only switch $(x, y) \mapsto (r, \theta)$ to polar, while keeping $z$ untouched. So everything reduces to

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

with $0 \leq \theta < 2\pi$ and $r \geq 0$. We have that $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$, as expected.

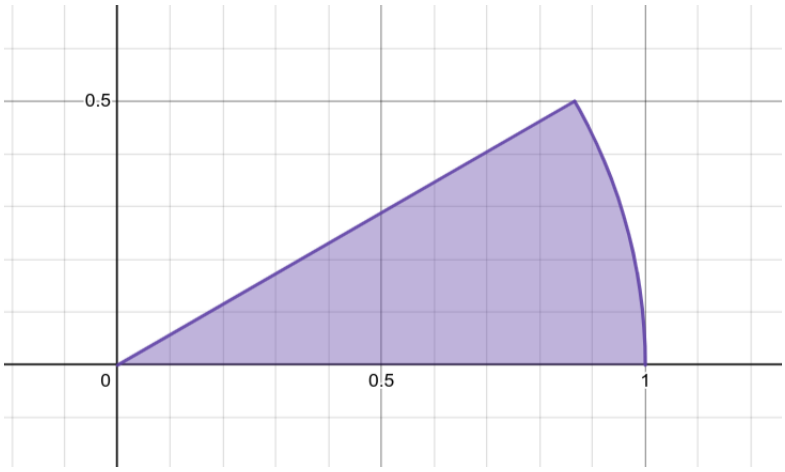
**Example 15**

*Describe the region of integration for* $\displaystyle\int_{-1}^{1}\int_{0}^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ *in cylindrical coordinates, and rewrite the integral appropriately.*

For each value of $z$ between $-1$ and 1, the bounds for the double integral

$$\int_{0}^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y$$

over the "$z$-slice" don't depend on $z$, which means that all slices look the same. The angle between the line of equation $x = \sqrt{3}y$ and the $x$-axis is $\pi/6$ (since $\tan(\pi/6) = \sqrt{3}/3 = 1/\sqrt{3}$ and $y = (1/\sqrt{3})y$), and we have that:




To wit, for each fixed value of $y$ between 0 and $1/2$, $x$ *starts* at $\sqrt{3}y$ and *stops* at $\sqrt{1-y^2}$. We have that

$$\int_{-1}^{1}\int_{0}^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{0}^{2\pi}\int_{0}^{\pi/6}\int_{0}^{1} f(r,\theta,z)r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

## 5.2 Spherical coordinates

**Idea:** In polar coordinates, we were describing a point $(x, y)$ with one radial parameter $r$ and one angle $\theta$. Now we will describe a point $(x, y, z)$ with a radial parameter $\rho$, but **two** angles $\theta$ and $\varphi$.

More precisely, suppose that $(x, y, z)$ is a point in space, and consider the following setup:

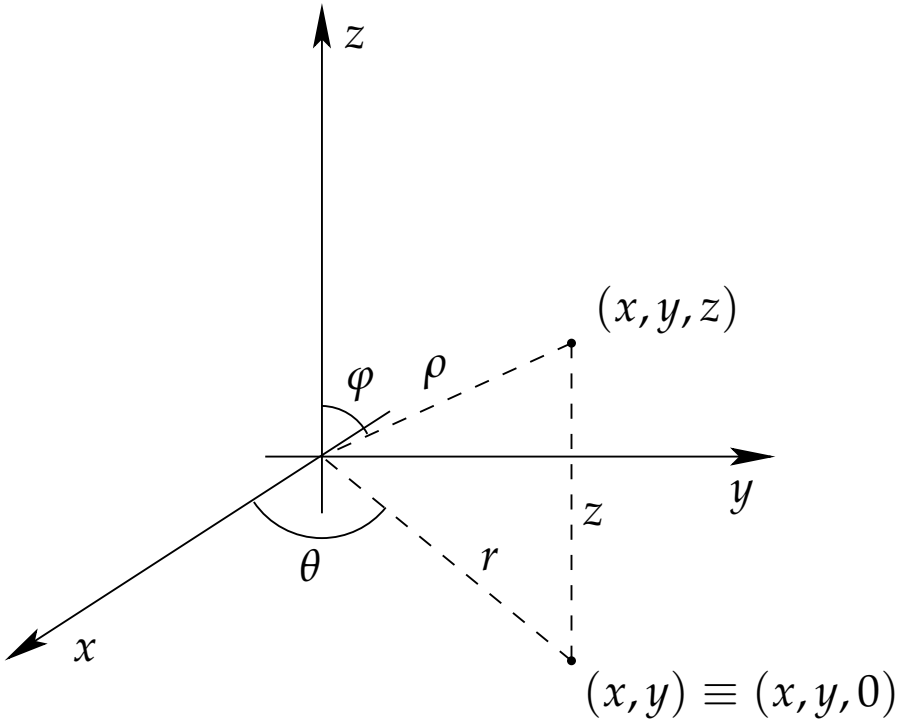



Figure 4: Spherical coordinates in $\mathbb{R}^3$.

Thinking about polar coordinates in the $xy$-plane, we may write $x = r\cos\theta$ and $y = r\sin\theta$. To relate $r$ with $\rho$ and $\varphi$, see the following right triangle:

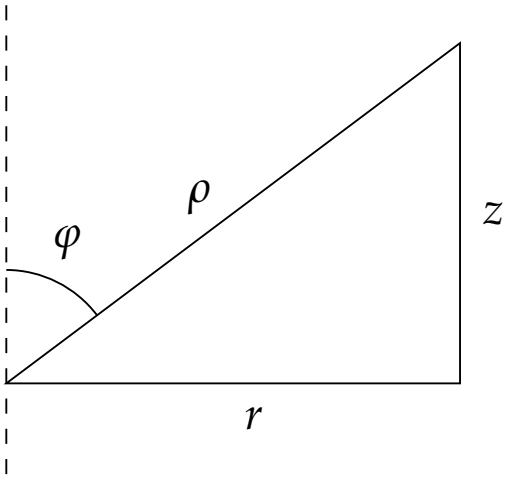



Figure 5: Eliminating the variable $r$.

Since we have $r = \rho \sin\varphi$ and $z = \rho\cos\varphi$, we may substitute it to obtain

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi. \end{cases}$$

We also note that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$, by the Pythagorean Theorem applied twice. To cover all of space, we must have

$$\rho \geq 0, \quad 0 \leq \theta < 2\pi \quad \text{and} \quad 0 \leq \varphi < \pi.$$

Observe that if $\varphi$ took values bigger than $\pi$, we would be counting some points in space twice. The correction between differentials is $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$. Let's put all of it together in an example:

**Example 16**

*Compute* $\displaystyle\iiint_D \frac{\mathrm{d}V}{(x^2+y^2+z^2)^{3/2}}$, *where* $D$ *is the solid bounded by the spheres of radius* 1 *and* 2 *centered at the origin.*

We directly have that

$$1 \leq \rho \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad \text{and} \quad 0 \leq \varphi \leq \pi.$$

This is because the smallest distance from a point in $D$ to the origin is 1, while the largest is 2. We are not omitting any directions around the $z$-axis, so the interval for $\theta$ is full and, lastly, $\varphi$ goes from 0 to $\pi$ (because stopping at $\pi/2$ would cover only the upper half of $D$). With this in place, we proceed:

$$\begin{aligned} \iiint_D \frac{\mathrm{d}V}{(x^2+y^2+z^2)^{3/2}} &= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{\rho^2\sin\varphi}{(\rho^2)^{3/2}}\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{\sin\varphi}{\rho}\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\ &\overset{(*)}{=} \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_1^2\frac{\mathrm{d}\rho}{\rho} \\ &= 2\pi\cdot 2\cdot\ln 2 \\ &= 4\pi\ln 2. \end{aligned}$$

Note that the step $(*)$ is only valid because all the bounds of integration before it were constants and the function being integrated was the product between a function only of $\rho$ with a function only of $\varphi$ (no mingling).

## 5.3 Changing variables

**Idea:** It is essentially a multivariable version of an $u$-substitution. Then, we had that if $u = g(x)$, then $\mathrm{d}u = g'(x)\,\mathrm{d}x$, and then

$$\int_a^b f(u)\,\mathrm{d}u = \int_c^d f(g(x))g'(x)\,\mathrm{d}x,$$

where $g(c) = a$ and $g(d) = b$. Two things happened here: we needed to update the interval of integration, and pay attention to the correction in the differential. The same thing will happen with more than one variable, with the only difference being that computing this correction is more complicated. Roughly, what happens is that

$$\mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \quad \text{and} \quad \mathrm{d}u\,\mathrm{d}v = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \mathrm{d}x\,\mathrm{d}y.$$

We will occasionally need to pay attention to both. Here, we have that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix},$$

and same for $\partial(u,v)/\partial(x,y)$. The bars here stand for "taking the absolute value of the determinant". How can you be sure you're not mixing things up? The following illegal cancellations should happen (in your head, never in the paper):

$$\mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \cancel{\mathrm{d}u\,\mathrm{d}v} \quad \text{and} \quad \mathrm{d}u\,\mathrm{d}v = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \cancel{\mathrm{d}x\,\mathrm{d}y}.$$

An expression like

$$\mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \mathrm{d}u\,\mathrm{d}v,$$

where things "don't want to cancel each other", is **wrong**. In any case, it is often easier to compute one of $|\partial(u,v)/\partial(x,y)|$ or $|\partial(x,y)/\partial(u,v)|$ over the other (depending on how the change of variables was set up or given to you). The strategy here is to always go for the easier one and use the inverse relation

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

if needed. Let's put all of this together in one last example, emphasizing all the steps to be carried out.

**Example 17**

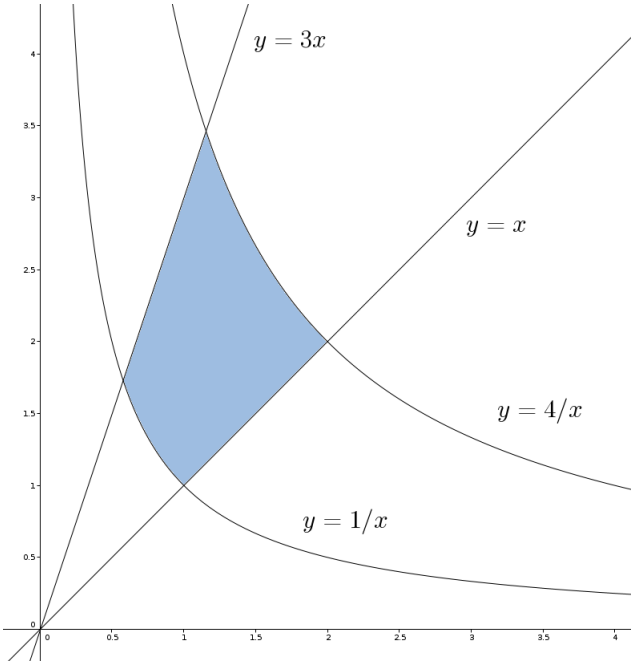
*Compute the integral*

$$\iint_R \mathrm{e}^{xy}\,\mathrm{d}A,$$

*where $R$ is the region in the first quadrant bounded by the hyperbolas $y = 1/x$ and $y = 4/x$, and the lines $y = x$ and $y = 3x$.*

- Step 1: sketch the region $R$. You just have to remember that the graph of

$y = 1/x$ is a branch of a hyperbola and that the 4 in $y = 4/x$ is just a rescaling factor.




- Step 2: rewrite the bounds of $R$ in the form "something" = "constant" and hopefully read the new variables and their bounds from there. Namely, we have that

$$\begin{array}{rcl} y = 1/x & \to & xy = 1 \\ y = 4/x & \to & xy = 4 \\ y = x & \to & y/x = 1 \\ y = 3x & \to & y/x = 3 \end{array}$$

  This suggests letting $u = xy$ and $v = y/x$. Immediately, the new bounds are $1 \leq u \leq 4$ and $1 \leq v \leq 3$. Who is $u$ and who is $v$ is not relevant here: the rectangle in the $uv$-plane corresponding to $R$ will just come out rotated, and the negative sign you get in the Jacobian determinant disappears because of the absolute value present when we write $\mathrm{d}u\,\mathrm{d}v$ in terms of $\mathrm{d}x\,\mathrm{d}y$ or vice-versa.

- Step 3: compute the Jacobian determinant. As we have $u$ and $v$ in terms of $x$ and $y$, it's easier to begin with

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} = \frac{y}{x} - x\left(-\frac{y}{x^2}\right) = 2\frac{y}{x} = 2v.$$

  As $v \geq 0$, we have $|2v| = 2v$, so

$$\mathrm{d}u\,\mathrm{d}v = 2v\,\mathrm{d}x\,\mathrm{d}y \implies \mathrm{d}x\,\mathrm{d}y = \frac{1}{2v}\,\mathrm{d}u\,\mathrm{d}v.$$

- Step 4: plug everything into the original integral and solve it.

$$\iint\limits_R \mathrm{e}^{xy}\,\mathrm{d}A = \int_1^3 \int_1^4 \frac{\mathrm{e}^u}{2v}\,\mathrm{d}u\,\mathrm{d}v = \left(\int_1^3 \frac{1}{2v}\,\mathrm{d}v\right)\left(\int_1^4 \mathrm{e}^u\,\mathrm{d}u\right) = \frac{\ln 3}{2}(\mathrm{e}^4 - \mathrm{e}).$$

# 6 September 26th

Free review before the first midterm.

# 7 October 3rd

## 7.1 Linear systems and RREF for matrices

We move on to the second part of this course: linear systems and matrices. The goal here is to understand the general mechanism to solve linear systems with several equations and variables (as opposed to small systems with only two or three equations and variables). The main tool we'll use to do so will be matrices. The program we'll try to carry out here is to:

$$\begin{aligned}\text{start with linear system} &\implies \text{convert it to a matrix}\\ &\implies \text{get a simpler matrix} \implies \text{get a simpler system,}\end{aligned}$$

and then draw conclusions about the original system from the original one. To understand how to convert between systems and matrices (that is, the first and last arrows above), let's consider the next two exercises. We'll address the middle arrow and explain what we mean by a "simpler matrix" soon.

**Example 18**

*What are the coefficient matrix and the augmented matrix for each of the following linear systems?*

$$\text{(a)}\begin{cases} x_1 + 3x_2 - x_3 = 1\\ 2x_1 + 5x_2 + x_3 = 5\\ x_1 + x_2 + x_3 = 3\end{cases} \qquad \text{(b)}\begin{cases} x_1 + x_2 + x_3 - x_4 = 1\\ -x_1 + x_2 - x_3 + x_4 = 3\\ -2x_1 + x_2 + x_3 - x_4 = 2\end{cases}$$

---

For item (a) we have that the coefficient and augmented matrices are, respectively,

$$\begin{bmatrix} 1 & 3 & -1\\ 2 & 5 & 3\\ 1 & 1 & 1\end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 3 & -1 & 1\\ 2 & 5 & 3 & 5\\ 1 & 1 & 1 & 3\end{array}\right].$$

Note that we have one row for each equation, and one column per variable in the coefficient matrix. This is a general phenomenon. To obtain the augmented matrix, we *augment* the coefficient matrix with an extra column, containing the right-hand-sides of the equations in the system. When there is no coefficient in front of a variable, as in $x_2$, it means $1x_2$. When dealing with augmented matrices, the last column is usually separated from the rest with a line or dotted line, just to remind us that it does not correspond to any variable. For item (b), we have that the the coefficient and augmented matrices are, respectively,

$$\begin{bmatrix} 1 & 1 & 1 & -1\\ -1 & 1 & -1 & 1\\ -2 & 1 & 1 & -1\end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1\\ -1 & 1 & -1 & 1 & 3\\ -2 & 1 & 1 & -1 & 2\end{array}\right].$$

How does the reverse process work?

**Example 19**

*If the following two matrices are the augmented matrices for two systems, write the systems explicitly.*

$$\text{(a)}\ \left[\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 1 \\ -2 & -3 & 5 & \pi & \sqrt{2} \\ 1/2 & 5 & 0 & 0 & 2023 \end{array}\right] \qquad \text{(b)}\ \left[\begin{array}{ccc|c} 0 & 1 & \mathrm{e} & -2 \\ 4 & \sqrt[3]{2} & -1 & 0 \\ 2 & 0 & \ln(5) & 3 \end{array}\right]$$

---

This time, we set up one equation per row, reading the coefficients in order, and recalling that the last column after the divide correspond to the right-hand-sides of the equations in the systems. For item (a) we obtain the system

$$\begin{cases} 2x_1 + 3x_2 - x_3 + x_4 & = 1 \\ -2x_1 - 3x_2 + 5x_3 + \pi x_4 & = \sqrt{2} \\ \frac{1}{2}x_1 + 5x_2 & = 2023, \end{cases}$$

while for item (b) we have that

$$\begin{cases} x_2 + \mathrm{e}x_3 & = -2 \\ 4x_1 + \sqrt[3]{2}x_2 - x_3 & = 0 \\ 2x_1 + \ln(5)x_3 & = 3. \end{cases}$$

There are two things to observe here.

- The first one is that while most examples you'll see on the book only have nice, integer coefficients, there is nothing saying that this must be the case. The word "linear" in linear system refers to the fact that each equation will be a "linear combination" of variables, as in the first equation

  $$2x_1 + 3x_2 - x_3 + x_4 = 1$$

  of the first system, but the numeric coefficients can be anything (such as $\pi$, $\mathrm{e}$, $\ln(5)$, $\cos(2)$, etc., all legit real numbers).

- The second one is that zeros in the coefficient matrix mean that the corresponding variables are simply absent from the corresponding equation. This is a *good thing*. Having fewer variables to worry about makes dealing with the system easier.

**Remark.** This is a convenient moment to introduce one piece of terminology: a linear system is called **homogeneous** if **all** the right-hand-sides of the equations in the system are equal to zero. For example, none of the systems presented so far in today's entry are homogeneous. It turns out that understanding homogeneous systems is slightly easier than understanding non-homogeneous one, and solutions to non-homogeneous systems can be found by first finding solutions to the "associated ho-

mogeneous system" (obtained by replacing all the right-hand-sides with zeros). More on this later. A similar idea will appear again when we move on to the third part of this class, discussing differential equations (they can be divided into homogeneous and non-homogeneous, and homogeneous ones are nicer).

Following up the idea of the second bulleted point above, we can return to the idea of going from the augmented matrix representing a system to a simpler matrix. The idea will be to create as many zeros as possible. More precisely, we will consider matrices in **reduced row echelon form (RREF)**, which looks like this:

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 & * & * \\ 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In other words, the three distinguishing features of a matrix in RREF are:

- the staircase pattern (*echelon*);
- all **pivots** (i.e., the first non-zero entry in each row) equal 1;
- only zeros above the pivots (*reduced*).

Going to RREF cannot be done in a haphazard way. For example, we must arrange for the three above features *in order*. In addition, how can one ensure that the RREF of a matrix will correspond to a system with the same solutions as the original one? To ensure that the solutions of the original system will not be affected, one casts a given matrix into RREF by using the following **elementary operations** are allowed:

(1) switch two rows;

(2) multiply any row by a non-zero number;

(3) add to any row a multiple of another row.

When solving a system, we will always follow this procedure of putting the **augmented** matrix into RREF (as opposed to the coefficient matrix), as doing operations between rows is morally the same as doing operations between equations in the system (with the obvious advantage that we won't keep writing variable names $x_1$, $x_2$, etc., all the time), and when doing operations between equations, those operations happen on the right-hand side of the equations involved too.

**Remark.** There are several ways of solving linear systems (one of them being "Cramer's method", for example), but dealing with matrices in RREF is preferred from a computational viewpoint for being more efficient, in the sense that solving a linear system using this algorithm is what takes a computer the fewer number of operations to do. Moreover, RREF is the answer to the natural question "could I do something else to make the matrix simpler?" or "am I missing something?". If the matrix is in RREF, the answer is "no": you did everything possible to simplify the matrix. And, again, *the reason why we'll only stick with the elementary operations above is because they make sure that the matrix in RREF obtained in the end does represent a system equivalent to the original one.*

**Example 20**

*Decide whether the following matrix is in RREF and, if not, put it in RREF.*

$$\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & -2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

---

The pivots of this matrix are, respectively, 1, 2 and 1. Since we have a pivot which is not equal to 1, the matrix is not in RREF. Each step we carry out has a specific objective, and whatever happens with the rest of the matrix, happens. Our first goal is to turn that pivot 2 into a 1, and for that we may divide the whole second row by 2 (this is an allowed elementary operation):

$$R_2 := \frac{1}{2}R_2 \qquad \sim \qquad \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note how we're borrowing notation from computer science to keep track of the operations performed. This is very important to do as it improves organization and readability. You should take this seriously on HW assignments and exams.

At this point, all that's left to do is to take out the trash above the pivots. This is done systematically, again from left to right, but now from bottom to top. The next goal should be to eliminate the 2 in the first row of the matrix.

$$R_1 := R_1 - 2R_2 \qquad \sim \qquad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since on the third column there is no pivot, nothing can be done about the non-zero entries there. We move on to clean up what's above the pivot on the fourth column. As both $R_1$ and $R_2$ will interact with $R_3$, but not with each other, we can do two steps at once:

$$\begin{matrix} R_1 := R_1 - R_3 \\ R_2 := R_2 + \frac{3}{2}R_3 \end{matrix} \qquad \sim \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have obtained the RREF.

Let's see how to put all of this together to solve a more complicated system.

**Example 21**

*Solve, if possible, the following system:*

$$\begin{cases} x_1 + x_2 \qquad\quad - x_5 = 1 \\ \quad x_2 + 2x_3 + x_4 + 3x_5 = 1 \\ x_1 \qquad - x_3 + x_4 + x_5 = 0 \end{cases}$$

---

Like in Example 18, let's set up the augmented matrix for this system:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{array}\right]$$

This matrix is obviously not in RREF; we don't even have the "staircase" shape. The first 1 in the first row, however, is the pivot, and so we need to eliminate all entries below it. This can be achieved as follows:

$$R_3 := R_3 - R_1 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & -1 & -1 & 1 & 2 & -1 \end{array}\right].$$

We have cleaned up everything below the first pivot, so we move on to the next. The pivot in the second row is the 1 appearing in the second column as well, so we need to eliminate the $-1$ below it. This is done by:

$$R_3 := R_3 + R_2 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Observe that this $-1$ could also have been eliminated via $R_3 := R_3 + R_1$, but this operation would produce a 1 in the bottom left corner of the matrix! The reason why this happened is because we're trying to make an operation involving a row whose pivot has already had everything cleaned up below it. The moral of the story here is that each step to be carried has a laser-like focus, a single goal, and if you do an operation which undoes something that should have been already ok by that point, you have done something wrong. We must now proceed to clean up the spaces above the pivots, going from left to right, as usual. Above the pivot in the second row, we have a 1, which is the next target:

$$R_1 := R_1 - R_2 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 0 & -2 & -1 & -4 & 0 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Observe that this operation did not destroy the "staircase" shape because we have only started to clean up spaces above the pivots once the matrix *already had the*

*staircase pattern completed*. We move on to the next pivot, on the third row: we must clean up the $-2$ and 2 above it. As in the previous example (and this turns out to be a general phenomenon), since both $R_1$ and $R_2$ will interact with $R_3$, but not with each other, we can do the two steps at once:

$$\begin{matrix} R_1 := R_1 + 2R_3 \\ R_2 := R_2 - 2R_3 \end{matrix} \quad \sim \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

The matrix is now in RREF, and it corresponds to the system

$$\begin{cases} x_1 \qquad\quad + 3x_4 + 6x_5 = 0 \\ \quad x_2 \qquad - 3x_4 - 7x_5 = 1 \\ \qquad\quad x_3 + 2x_4 + 5x_5 = 0, \end{cases}$$

which is equivalent to the original one (that is, this system and the first one have the same solution set). The variables $x_4$ and $x_5$ (corresponding to the columns in RREF which could not be cleaned up – as we didn't have pivots to use) are **free**, in the sense that the remaining variables $x_1$, $x_2$ and $x_3$ may be written in terms of $x_4$ and $x_5$. We may change notation, say, to $t_1 = x_4$ and $t_2 = x_5$, effectively **parametrizing the solution set of the system**, and writing

$$S = \{(-3t_1 - 6t_2, 3t_1 + 7t_2 + 1, -2t_1 - 5t_2, t_1, t_2) \in \mathbb{R}^5 \mid t_1, t_2 \in \mathbb{R}\}.$$

The solution set $S$ is a 2-dimensional plane in $\mathbb{R}^5$, not passing through the origin $(0,0,0,0,0)$ of $\mathbb{R}^5$. Every point of such plane corresponds to a solution of the system. For example, choosing $t_1 = 1$ and $t_2 = 2$, we obtain the solution $(-15, 18, -12, 1, 2)$ of the original system. This is the same as saying that plugging

$$x_1 = -15, \quad x_2 = 18, \quad x_3 = -12, \quad x_4 = 1 \quad \text{and} \quad x_5 = 2$$

on the equations of the original system, the right-hand sides come out to be 1, 1, and 0. Every time you have even a single free variable, the system has infinitely many solutions. These variables are called "free" because you're free to choose values to substitute into it, thus generating different solutions to the system. If there are infinitely many solutions (in fact, **uncountably many**, in a very precise sense), you are not supposed to try and list them one by one. This is why it is important to understand how free variables work and how to describe your solution set in a "parametric form": it will carry all the information you need in a succint way.

# 8 October 10th

## 8.1 A slightly more abstract problem

Recall that a linear system is called **compatible** if it has solutions, and **incompatible** if it has no solutions. A compatible linear system always has either a unique solution, of infinitely many solutions (there is no other possibility).

**Example 22**

*For each system, find the value(s) of $c$ for which they have no solution:*

$$\text{(a)} \begin{cases} x_1 + 3x_2 = 4 \\ 2x_1 + 6x_2 = c \end{cases} \qquad \text{(b)} \begin{cases} 3x_1 + cx_2 = 3 \\ cx_1 + 3x_2 = 5 \end{cases}$$

**Hint:** *Try to solve them and see which values of $c$ threaten a division by zero.*

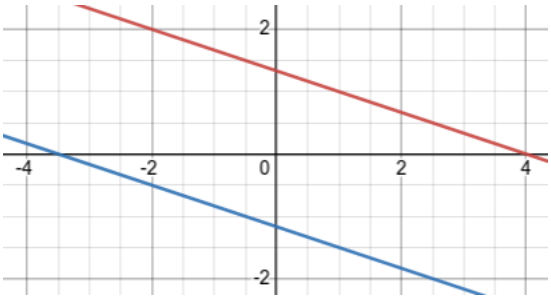
---

Let's treat both systems separately.

(a) Set up the augmented matrix and put it into RREF, by doing $R_2 := R_2 - 2R_1$:

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 6 & c \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 0 & c-8 \end{array}\right].$$

If $c - 8 = 0$ (i.e., if $c = 8$), the last matrix is already in RREF, and we have that the second column does not contain a pivot, meaning that $x_2$ is a free variable, and the system is compatible (having infinitely many solutions). If $c - 8 \neq 0$ (i.e., if $c \neq 8$), we'll have a row of the form $\left[\begin{array}{cc|c} 0 & 0 & \text{nonzero} \end{array}\right]$, which says that the system is incompatible. Geometrically, what happens is that in the $x_1x_2$-plane, the lines described by the equations $x_1 + 3x_2 = 4$ and $2x_1 + 6x_2 = c$ are always parallel, and hence equal or disjoint. They are equal **precisely** when $c = 8$, and disjoint otherwise. The solutions of the system are the intersections between the lines. See the following figure.




Play around with the values of $c$ yourself at

https://www.desmos.com/calculator/vxqc7v8lf2.

(b) Again, start setting up the augmented matrix as

$$\left[\begin{array}{cc|c} 3 & c & 3 \\ c & 3 & 5 \end{array}\right].$$

Normalize the first row to obtain a pivot of 1, and then eliminate the $c$ below it by doing $R_2 := R_2 - cR_1$:

$$\left[\begin{array}{cc|c} 1 & c/3 & 1 \\ c & 3 & 5 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & c/3 & 1 \\ 0 & 3 - \frac{c^2}{3} & 5 - c \end{array}\right].$$

Now, there are cases to look at. Observe that $3 - c^3/3 = 0$ is equivalent to having $c = \pm 3$. But for those values of $c$, we have that $5 - c \neq 0$, and so the system is incompatible due to the presence of a row of the form $\left[\begin{array}{cc|c} 0 & 0 & \text{nonzero} \end{array}\right]$ (as in the previous item). It is worth pointing out that if, instead of $5 - c$, we had some function of $c$ whose value were equal to zero for $c = 3$ (resp. $c = -3$), then $c = 3$ (resp. $c = -3$) would **not** make the system incompatible – to wit, the last matrix above would be in RREF, with the second column corresponding to a free variable. And if $c \neq \pm 3$, one may proceed to normalize the second row as to obtain something of the form

$$\left[\begin{array}{cc|c} 1 & * & * \\ 0 & 1 & * \end{array}\right],$$

which always corresponds to a compatible system (why?).

Geometrically, we observe that the lines in the $x_1x_2$-plane whose equations are $3x_1 + cx_2 = 3$ and $cx_1 + 3x_2 = 5$ are parallel precisely when $c = \pm 3$ and, in this case, they are disjoint. It's Desmos time again:

https://www.desmos.com/calculator/4ls7umhvtj.

## 8.2 Some matrix algebra

Let us do a quick review on matrix algebra. Namely, essentially everything you cand do with vectors, you can do with matrices. We can:

- add or subtract two matrices **of the same size**, as in

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 8 \\ 6 & 9 & 12 \end{bmatrix},$$

entrywise.

- rescale a matrix of any size (by multiplying it by a real number), as in

$$3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix},$$

again entrywise.

- multiply a $n \times m$ matrix $A$ by a $m \times k$ matrix $B$, to obtain a $n \times k$ matrix $AB$, whose $(i, j)$-entry equals the **dot product** between the $i$-th row of $A$ and the $j$-th column of $B$. Idea: $(n \times m)(m \times k) = n \times k$. If the sizes of the matrices are not "compatible" as indicated, we would be trying to compute the dot product between vectors of different dimensions, which makes no sense.

Here's a reason about why matrix multiplication is like that: it all comes back to the big idea of representing a given linear system with a matrix, studying the matrix instead, and drawing conclusions about the original system from, for example, the reduced row echelon form of the obtained matrix. More precisely, think of the silliest case possible, where we have only one equation and one variable: $ax = b$. If we want to study linear systems (say, with $n$ variables and $m$ equations) with sort of the same notation, $A\boldsymbol{x} = \boldsymbol{b}$, we need to make sense of what does it mean to multiply the $m \times n$ matrix $A$ with the $n \times 1$ column vector $\boldsymbol{x}$, to obtain the $m \times 1$ column vector $\boldsymbol{b}$. Write the system explicitly:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The entries of the product $A\boldsymbol{x}$ should be the entries of $\boldsymbol{b}$, but the system itself gives the expression fot the entries of $\boldsymbol{b}$ in terms of $A$ and $\boldsymbol{x}$. The definition

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

is thus **forced** upon us, in the sense that if we want to make sense of the system in the form $A\boldsymbol{x} = \boldsymbol{b}$, there is only one possible choice for what the vector $A\boldsymbol{x}$ must be. To multiply $A$ by a second matrix $B$ which is $n \times k$, one applies the above for each column of $B$ — and this is the matrix multiplication you have first seen in class. In particular, this should justify why the number of columns of $A$ must equal the number of rows of $B$ for this product to make sense.

Here are some pitfalls to avoid, regarding matrix multiplication:

- matrix multiplication is **non-commutative**, i.e., $AB$ is not equal to $BA$, in general; one way to convince yourself quickly of this is that unless both $A$ and $B$ are square matrices of the same size, in general only one of the products $AB$ or $BA$ is well-defined, while the other is not, so it doesn't even make sense to compare them (and to make it worse, there are examples of square matrices $A$ and $B$ for which $AB \neq BA$). This means that while one does $(a + b)^2 = a^2 + 2ab + b^2$ for real numbers $a$ and $b$, one must write $(A + B)^2 = A^2 + AB + BA + B^2$ in full, with no further simplifications available.

- one cannot "cancel" non-zero matrices like real numbers: if $AB = AC$ and $A \neq 0$, we **cannot conclude** that $B = C$. In fact, this only works provided that $A$ is non-singular (that is, $A$ is square and the only solution of $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \boldsymbol{0}$).

- the identity matrix (which plays the role of 1 for real numbers) is not the matrix full of 1's, but instead the matrix with 1's in the diagonal and zeros everywhere else.

To further emphasize what was explained above: it is not always true that any two matrices may be multiplied. Namely, the $(i,j)$-th entry of a product $AB$, whenever it makes sense, must be equal to the dot product between the $i$-th entry of $A$ and the $j$-th column of $B$. For this to make sense, such rows and columns **must have the same size**. Therefore, only products like $(n \times m)(m \times k)$ make sense, with output having size $n \times k$. Let's drill all of this with the next problem.

**Example 23** (Matrix multiplications may not "compile")

*Consider the three matrices*

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 6 & 1 & 3 & 5 \\ 2 & 4 & 2 & 0 \end{bmatrix}.$$

*Among all $3^2 = 9$ products $A^2$, $AB$, $BA$, $AC$, $CA$, $B^2$, $BC$, $CB$, and $C^2$, decide which ones are well-defined and which ones are not. Evaluate the ones which are well-defined.*

---

Let's look at it systematically, thinking that $A$ has size $3 \times 2$, $B$ has size $3 \times 3$, and $C$ has size $3 \times 4$.

- $A^2$: we have $(3 \times 2)(3 \times 2)$, so it's undefined.
- $AB$: we have $(3 \times 2)(3 \times 3)$, so it's undefined.
- $AC$: we have $(3 \times 2)(3 \times 4)$, so it's undefined.
- $BA$: we have $(3 \times 3)(3 \times 2)$, so $BA$ is defined and has size $3 \times 2$. We have that
$$BA = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 21 \\ 43 & 53 \\ 20 & 12 \end{bmatrix}.$$
- $CA$: we have $(3 \times 4)(3 \times 2)$, so it's undefined.
- $B^2$: we have $(3 \times 3)(3 \times 3)$, so $B^2$ is defined and has size $3 \times 3$. We have that
$$B^2 = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 10 & 8 \\ 53 & 30 & 22 \\ 12 & 12 & 7 \end{bmatrix}.$$

- $BC$: we have that $(3 \times 3)(3 \times 4)$, so $BC$ is defined and has size $3 \times 4$. We have that

$$BC = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 & 0 \\ 6 & 1 & 3 & 5 \\ 2 & 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 12 & 10 \\ 44 & 23 & 46 & 20 \\ 14 & 10 & 26 & 0 \end{bmatrix}.$$

- $CB$: we have $(3 \times 4)(3 \times 3)$, so it's undefined.
- $C^2$: we have $(3 \times 4)(3 \times 4)$, so it's undefined.

## 8.3 Linear independence

The last concept we need to discuss in this part of the course is the one of **linear independence**. Geometrically in low dimensions, we have that:

- Two vectors in the plane $\mathbb{R}^2$ are linearly independent if they are not collinear.
- Two vectors in space $\mathbb{R}^3$ are linearly independent if they are not collinear.
- Three vectors in space $\mathbb{R}^3$ are linearly independent if they are not coplanar, or not all collinear.

How to make sense of this in higher dimensions, where pictures are no longer available?

**Definition 2**

Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^n$ be vectors. We say that they are **linearly independent** if whenever $a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k = \boldsymbol{0}$ (here, $a_1, \ldots, a_k$ are real numbers), we must necessarily have $a_1 = \cdots = a_k = 0$. If they are not linearly independent, we call them **linearly dependent**.

Here is one example: the vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

in three-dimensional space $\mathbb{R}^3$ are linearly independent. This is because whenever $a_1, a_2, a_3$ are real, the condition

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

reads

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and so $a_1 = a_2 = a_3 = 0$. So, you can think of linearly independent vectors as being in "general position" as the standard vectors above are (although the fact that they are orthogonal here is not immediately relevant – linear independence is an algebraic notion, not geometrically).

Another way to see is is by saying that a collection of vectors is linearly independent is saying that none of the vectors considered is a linear combination of the others. Similarly, saying that a collection of vectors is linearly dependent is saying that there is at least one of them which may be expressed as a linear combination of the others.

How to make sense of the condition $a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k = \boldsymbol{0} \implies a_1 = \ldots = a_k = 0$ in terms of what we already know? Note that $a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k = \boldsymbol{0}$ is, in disguise, a homogeneous linear system for the unknowns $a_1, \ldots, a_k$! Linear independence means that such system has only the trivial solution. For example, in the above situation the system was trivial:

$$\begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$$

In the general case, this system can be written in vector form as

$$[\boldsymbol{u}_1|\cdots|\boldsymbol{u}_k]\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $[\boldsymbol{u}_1|\cdots|\boldsymbol{u}_k]$ is the matrix whose columns are $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$. This suggests the following **algorithm** for deciding when a given collection of vectors is linearly independent:

1. Set the given vectors as columns of a matrix.
2. Put it into RREF, or at least reduce it to "triangular" form, to identify the **rank**, i.e., the number of pivots. (In fact, RREF is overkill: you can identify the rank as soon as you have obtained a staircase pattern.)
3. If the number of pivots is less than the number of vectors considered (which happens whenever you get a row of zeros), they are linearly dependent[3]. If the number of pivots equals the number of vectors considered, they are linearly independent. You don't have to worry about the number of pivots being greater than the number of vectors considered, since this is impossible to happen (why?).

Let's practice the algorithm in practice with the next problem:

**Example 24**

Consider the (column) vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \boldsymbol{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

[3] In more detail: since there is a free variable, you get nontrivial solutions for $a_1, \ldots, a_k$, which correspond to a nontrivial linear combination of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ resulting in $\boldsymbol{0}$.

(a) Are $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ linearly independent?

(b) Are $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, and $\boldsymbol{u}_4$ linearly independent?

We start with (a). Applying the algorithm described above. Namely, we consider the matrix

$$[\boldsymbol{u}_1|\boldsymbol{u}_2|\boldsymbol{u}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ -1 & -3 & 3 \end{bmatrix}.$$

Then

$$\begin{matrix} R_2 := R_2 - 2R_1 \\ R_3 := R_3 + R_1 \end{matrix} \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & -1 & 2 \end{bmatrix}.$$

Now

$$R_2 := -\frac{1}{3}R_2 \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix},$$

and so

$$R_3 := R_3 + R_2 \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the rank equals 2, and since $2 < 3$, the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are linearly **dependent**.

For item (b), the same idea works. To make the calculations a bit easier, we note that the order of the vectors is completely irrelevant in the definition of linear independence (which makes perfect sense, as the rank of a matrix is not affected by permuting columns), so we may just as well place $\boldsymbol{u}_4$ as the first vector and set up

$$[\boldsymbol{u}_4|\boldsymbol{u}_2|\boldsymbol{u}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 4 \\ 0 & -3 & 3 \end{bmatrix}.$$

Then

$$R_2 = R_2 - R_1 \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & -3 & 3 \end{bmatrix},$$

and next

$$R_3 := R_3 - 3R_2 \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 12 \end{bmatrix}.$$

This last matrix is in triangular form, and has three pivots (not all equal to 1, though). This means that the matrix has full rank, and so $\boldsymbol{u}_2, \boldsymbol{u}_3$ and $\boldsymbol{u}_4$ are linearly **independent**.

# 9 October 17th

Free review before the second midterm. Let us just register a few core ideas, useful for discussing the number of solutions of a linear system without actually having to solve it.

1) If a linear system has more variables than equations, then it must have a free variable. The idea here is that, roughly, each equation will constrain one variable, so if there are more variables than equations, there are not enough equations to constrain all variables.

2) If a consistent linear system has a free variable, then it must have infinitely many solutions.

3) Homogeneous linear systems are always consistent. This is because they always admit the trivial solution $(x_1, \ldots, x_n) = (0, \ldots, 0)$. Think of the entire string "trivial solution" as being a name, not an adjective. This solution consisting only of zeros, working for every homogeneous system, is special and therefore gains this special name.

4) If a linear system with the same number of equations and variables, say $n$, has its coefficient matrix having full rank $n$, then the system has a unique solution.

As a consequence of the first three points above, we also have that a homogeneous linear system with more variables than equations automatically has infinitely many solutions. In addition, note that the assumption of consistency in the second point is crucial: the system whose augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 4 & 6 & 10 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right]$$

has three equations and four variables, and so it must have a free variable (in this case you can immediately see that $x_2$ and $x_4$ are free, but this is just because the matrix already happens to almost present the staircase pattern). However, the presence of a free variable does **not** imply that such system has infinitely many solutions: to apply the second point above, you need to know **beforehand** that the system is consistent. The system in fact turns out to be inconsistent due to the second and third equations $x_3 + 2x_4 = 1$ and $x_3 + 2x_4 = 2$ leading to $1 = 2$.

Alternatively, once you know that a system has free variables, to decide whether it is consistent or not, you can perform the following test: set all the free variables equal to zero, and see if the resulting system for the unconstrained variables is consistent. In the above situation, setting $x_2 = x_4 = 0$, we are led to

$$\left[\begin{array}{cc|c} 1 & 6 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right],$$

which again is manifestly inconsistent (the second constrained variable would simultaneously be equal to 1 and 2).

# 10 October 24th

## 10.1 Ordinary Differential Equations

We now start with the third part of this course: **Ordinary Differential Equations** ("ODE"s, for short). The idea is simple: we have equations involving derivatives, but instead of solving for a number or vector, we solve for a function instead. Consider a simple example: $y'(t) = 2t$, and imagine you want to solve for $y$. Integrating, we obtain that $y(t) = t^2 + c$ for some constant of integration $c \in \mathbb{R}$. Keeping track of such constants is now crucial (and perhaps the reason why you kept losing points in Calculus 1 for forgetting the $+c$; it was all a prelude to this moment), for different choices of $c$ lead to different solutions. As we have infinitely many choices of $c$, we see that even the simplest differential equation will have infinitely many solutions.

The adjective "ordinary" refers to the fact that all the functions involved are functions of a single variable, and that there are no partial derivatives of anything in play. If this were to be the case, we would be dealing with **Partial Differential Equations** ("PDE"s, for short) instead.

In any case, the point remains that solving ODEs can be very hard and one usually resorts to softwares or numerical methods to understand the behavior of solutions to an ODE. We will focus on very specific types of ODEs which we can indeed solve. To understand when this is the case, some vocabulary is useful. Let's always organize our equations by placing in the left side all the terms involving $y$, and on the right side all the terms not involving $y$.

- **order:** the order of a differential equation is the highest derivative that appears.
- **linearity:** a differential equation is linear if its left side is a linear combination (with function coefficients) of $y$, $y'$, $y''$, etc.
- **homogeneity:** a differential equation is homogeneous if its right side equals zero.

We care about this because: the higher the order, the harder the equation should be to solve; we like linear things better than non-linear things, and homogeneous equations are generally easier to deal with (because we like zeros).

Let's practice the terminology:

**Example 25**

Classify the following differential equations (order/linearity/homogeneity):

(a) $y'' - 4y' + 2y = 10t^2$.

(b) $y' - 2y^3 = -4t$.

(c) $y''' - \cos(t)y'' + e^t y = 0$.

---

(a) It's a second order equation because the term with the highest derivative is $y''$. It is linear because the left side is a linear combination of $y$, $y'$ and $y''$ (with

coefficients 2, $-4$ and 1, respectively). It is non-homogeneous as the right side does not equal the zero function.

(b) It's a first order equation because the term with the highest derivative is $y'$. It is non-linear because of the $y^3$ term (the non-linear operation of taking a cube is being applied to the function variable $y$). It is non-homogeneous for hopefully obvious reasons.

(c) It's a third order equation because the term with the highest derivative is $y'''$. It is linear because the left side is a linear combination of $y$, $y'$, $y''$ and $y'''$ (with coefficients $e^t$, $0$, $-\cos(t)$ and $1$, respectively). It is clearly homogeneous. Note here that the functions $\cos(t)$ and $e^t$ are nonlinear functions, but of the variable $t$, not the variable $y$ (which is what we care about when deciding whether a given differential equation is linear or not).

**Remark.** Warning: choosing particular values of $t$ does not tell you anything about the nature of the differential equation. For example, saying that the equation in item (a) is homogeneous for $t = 0$ and nonhomogeneous for $t \neq 0$ is **nonsense**. What matters is if what's on the right side is the **zero function** or not.

## 10.2 2nd order homogeneous linear ODEs with constant coefficients

Next, we will focus on **second order linear homogeneous ODEs, with constant coefficients**. This means, for instance, that the ideas we will present next are good for trying to solve an equation whose left side looks like the one in item (a) of the previous exercise, but not something like in item (c) (which, as a matter of fact, is very difficult to solve). The reason why we are going straight to second order equations, as opposed to first order ones, is that first order equations are "easy", in the sense that morally, all one has to do is one integration. And this is not enough when dealing with second order equations.

To further motivate what will come next, consider the simplest equation ever: $y'(t) = y(t)$. The solutions are clearly $y(t) = ce^t$, for some constant $c \in \mathbb{R}$. If we considered $y'(t) = ry(t)$ instead, the solutions would be $y(t) = ce^{rt}$. There is no reason for us to expect exponentials to come again to the rescue in the second order case, so we will think backwards. The idea is to **try** an exponential $y = e^{rt}$, and find the values of $r$ for which this is indeed a solution. Namely, $y' = re^{rt}$ and $y'' = r^2e^{rt}$, so

$$0 = ay'' + by' + cy = ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} \implies ar^2 + br + c = 0,$$

as $e^{rt} \neq 0$ may be cancelled.

So, we can solve the differential equation provided we can solve the **characteristic equation** $ar^2 + br + c = 0$. Note that this idea works even if the order of the differential equation is bigger than two, but then solving the characteristic equation may be difficult. When solving a quadratic equation, only three possibilities may happen:

**Theorem 2**

Consider the differential equation $ay'' + by' + cy = 0$, with $a, b, c \in \mathbb{R}$, and $a \neq 0$. If $r_1$ and $r_2$ are the solutions of the **characteristic equation** $ar^2 + br + c = 0$, then:

(i) if $r_1$ and $r_2$ are both real and distinct, the general solution of the given ODE is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, with $c_1, c_2 \in \mathbb{R}$.

(ii) if $r := r_1 = r_2$ is a real double root, the general solution of the given ODE is $y = c_1 e^{rt} + c_2 t e^{rt}$, with $c_1, c_2 \in \mathbb{R}$.

(iii) if $r_1$ and $r_2$ are complex (and hence conjugate to each other[a]), the general solution of the given ODE is $y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$, with $c_1, c_2 \in \mathbb{R}$, where $r_1 = \alpha + i\beta$.

[a] Complex roots of a real polynomial always come in conjugate pairs.

Of course, memorizing the above is a waste of time. Working case by case from here on saves you brain power. On the following examples, we will simultaneously explore this idea, as well as the fact that once suitable initial conditions have been imposed, the solution to the so-called **Initial Value Problem** ("IVP", for short) becomes unique.

**Example 26** (Two distinct real roots)

(a) *Find the general solution of* $y'' - 3y' - 18y = 0$.

(b) *Find the unique solution with initial conditions* $y(0) = 0$ *and* $y'(0) = 4$.

---

For item (a), we start setting up the characteristic equation $r^2 - 3r - 18 = 0$. It may be factored as $(r-6)(r+3) = 0$, which says that the characteristic roots are $r_1 = 6$ and $r_2 = -3$. Therefore, we know that $y_1 = e^{6t}$ and $y_2 = e^{-3t}$ are two solutions. They are **linearly independent** because one is not a (real) multiple of the other. However, **since the given ODE is linear and homogeneous, the dimension of the space of solutions equals the order of the equation**. This says that taking linear combinations of $y_1$ and $y_2$ does, in fact, produce all solutions of this ODE. In other words, the general solution is

$$y = c_1 e^{6t} + c_2 e^{-3t}, \text{ with } c_1, c_2 \in \mathbb{R}.$$

For item (b), imposing **two** initial conditions (at the same point) allows us to solve for the **two** coefficients $c_1$ and $c_2$. The relations $y(0) = 0$ and $y'(0) = 4$ becomes the linear system

$$\begin{cases} c_1 + c_2 = 0 \\ 6c_1 - 3c_2 = 4 \end{cases} \implies c_1 = \frac{4}{9} \quad \text{and} \quad c_2 = -\frac{4}{9}.$$

The solution of the given IVP is

$$y = \frac{4}{9}e^{6t} - \frac{4}{9}e^{-3t}.$$

Let's see next what happens in the case where two complex conjugate roots appear. Two additional facts are crucial to understand this case:

1. Real and imaginary parts of a **complex solution** to a **real** linear homogeneous ODE are **real solutions**.

2. For any real number $\theta$, we have $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula).

We move on (phrasing things in a lazier way, but with the same content as in Example 26):

**Example 27** (Complex conjugate roots)

Solve the IVP:

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 8, \quad y'(0) = -8 \end{cases}$$

---

The strategy will always be the same: first find the general solution of the differential equation alone, and then use the given initial conditions to solve for $c_1$ and $c_2$. The characteristic equation is simply $r^2 + 9 = 0$, whose roots are $\pm 3i$. We may just focus on one of them[a], say $3i$. This says that $e^{3it}$ is a complex solution. By Euler's Formula,

$$e^{3it} = \cos(3t) + i\sin(3t),$$

and so $y_1 = \cos(3t)$ and $y_2 = \sin(3t)$ are real solutions. They're clearly linearly independent, so the general solution of the differential equation is

$$y = c_1\cos(3t) + c_2\sin(3t), \quad c_1, c_2 \in \mathbb{R}.$$

With this in place, we move on to impose the initial conditions. Computing the derivative as $y' = -3c_1\sin(3t) + 3c_2\cos(3t)$, we see that $y(0) = 8$ and $y'(0) = -8$ together give us

$$c_1 = 8 \quad \text{and} \quad 3c_2 = -8,$$

so the unique solution to the IVP is

$$y = 8\cos(3t) - \frac{8}{3}\sin(3t).$$

[a] Because complex roots of a real polynomial always come in conjugate pairs. So $3i$ and $-3i$ carry the same amount of information about the original differential equation.

There is one last case to study.

**Example 28** (Real double root)

Solve the IVP:

$$\begin{cases} y'' - 2y' + y = 0 \\ y(0) = 4, \quad y'(0) = 0 \end{cases}$$

---

The characteristic equation of the given ODE in this case is just $r^2 - 2r + 1 = 0$, so that $(r-1)^2 = 0$ says that $r = 1$ is a real double root. Hence $y_1 = \mathrm{e}^t$ is one solution, but since the equation has order 2, we need a second solution $y_2$, linearly independent from $y_1$, to span the whole solution space via linear combinations. Indeed, repeating $y_1$ and writing $c_1\mathrm{e}^t + c_2\mathrm{e}^t$ just leads to $(c_1 + c_2)\mathrm{e}^t$, but $c_1 + c_2$ is as arbitrary as $c_1$ and $c_2$, so it really counts as one single degree of freedom. One attempt to create linear independence is to replace $c_1 + c_2$ with $c_1 + c_2 t$, which suggests that $y_2 = t\mathrm{e}^t$ works as a second solution independent from $y_1$ and that the general solution is

$$y = c_1\mathrm{e}^t + c_2 t\mathrm{e}^t, \quad c_1, c_2 \in \mathbb{R}.$$

This is indeed the case by Theorem 2. Imposing initial conditions works the same as the other cases, so we may compute $y' = c_1\mathrm{e}^t + c_2\mathrm{e}^t + c_2 t\mathrm{e}^t$ by the product rule. Thus, $y(0) = 4$ and $y'(0) = 0$ together read as the system

$$\begin{cases} c_1 = 4 \\ c_1 + c_2 = 0 \end{cases} \implies c_1 = 4 \quad \text{and} \quad c_2 = -4,$$

and the unique solution to the IVP is

$$y = 4\mathrm{e}^t - 4t\mathrm{e}^t.$$

## 10.3 A higher order characteristic equation

As mentioned before, this mechanism can be used to solve linear homogeneous ODEs with constant coefficients of **any order**, say

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \quad \text{with } a_1, \ldots, a_n \in \mathbb{R},\ a_n \neq 0,$$

provided one can solve the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

instead. Once this equation has been completely factored, one builds the general solution from the factors obtained, using what has been discussed so far. Let's illustrate this with one more complicated last example.

**Example 29** (Dealing with higher order)

*Determine the general solution of a 5th order linear homogeneous ordinary differential equation whose characteristic equation is factored as*

$$(r-2)(r-3)^2(r-(4+5\mathrm{i}))(r-(4-5\mathrm{i}))=0.$$

---

Let's understand each factor separately:

- The term $(r-2)$ provides $y_1 = \mathrm{e}^{2t}$.
- The term $(r-3)^2$ provides $y_2 = \mathrm{e}^{3t}$ and $y_3 = t\mathrm{e}^{3t}$.
- The term $(r-(4+5\mathrm{i}))$ provides the complex solution $\mathrm{e}^{(4+5\mathrm{i})t}$, so using Euler's formula to write

$$\mathrm{e}^{(4+5\mathrm{i})t} = \mathrm{e}^{4t}\mathrm{e}^{5\mathrm{i}t} = \mathrm{e}^{4t}(\cos(5t)+\mathrm{i}\sin(5t)) = \mathrm{e}^{4t}\cos(5t)+\mathrm{i}\mathrm{e}^{4t}\sin(5t)$$

gives us the real solutions $y_4 = \mathrm{e}^{4t}\cos(5t)$ and $y_5 = \mathrm{e}^{4t}\sin(5t)$.

We conclude that the general solution is

$$y = c_1\mathrm{e}^{2t}+c_2\mathrm{e}^{3t}+c_3t\mathrm{e}^{3t}+c_4\mathrm{e}^{4t}\cos(5t)+c_5\mathrm{e}^{4t}\sin(5t),$$

with $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$.

If you want an extra reference for these things, I particularly like Chapters 3 and 4 of [2].

# 11 October 31st

## 11.1 The non-homogeneous case

As a follow up from last class, we now move on to discuss **non-homogeneous** second order linear ODE's with constant coefficients. Namely, we consider

$$ay'' + by' + cy = f,$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$ (or else the equation would be of first order instead), but with the right side being equal to some arbitrary function $f$. Here's what we need to know about it:

**Theorem 3**

The general solution of $ay'' + by' + cy = f$ is $y = y_p + y_h$, where $y_h$ is the general solution of the associated homogeneous equation $ay'' + by' + cy = 0$, and $y_p$ is any particular solution of the original non-homogeneous equation.

In other words, to solve such a non-homogeneous equation, we first consider its homogeneous version and find $y_h$, by methods already studied. As for finding $y_p$, however, the answer is dissapointing: we have to guess it. Of course, by "guess" we mean a "reasonable" guess, in the sense that if $f$ is a polynomial, trigonometric function, exponential, etc., we'll try to find $y_p$ of the same type. This method of guessing **almost** always works. We will explore this in the next examples, and also see how could guessing go wrong.

**Example 30**

Determine the general solution of the non-homogeneous linear differential equation $y'' - 9y = 2t + 1$.

---

Let's organize our work in steps:

1. Finding $y_h$: we need to find the general solution $y_h$ of the associated homogeneous equation $y'' - 9y = 0$. The characteristic equation is simply $r^2 - 9 = 0$, which may be factored as $(r-3)(r+3) = 0$. As this characteristic equation appeared by looking for the values of $r$ for which $e^{rt}$ was actually a solution of the homogeneous equation, we obtain two linearly independent solutions $e^{3t}$ and $e^{-3t}$, so that $y_h = c_1 e^{3t} + c_2 e^{-3t}$, with $c_1, c_2 \in \mathbb{R}$.

2. Finding $y_p$: as $2t + 1$ is a polynomial of degree 1, we try to make $y_p = At + B$ a polynomial of degree 1 as well. The goal is to find $A$ and $B$ for which $y_p$ is actually a solution of the original non-homogeneous equation. Plugging it into the differential equation, we have that $y_p'' - 9y_p = -9y_p = -9At - 9B = 2t + 1$, so that $A = -2/9$ and $B = -1/9$. The particular solution is $y_p(t) = -2t/9 - 1/9$.

3. Putting everything together: the general solution to the original non-homogeneous is
$$y(t) = -\frac{2}{9}t - \frac{1}{9} + c_1 e^{3t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

**Remark.** At this point, it is worth pointing out that if you were asked to solve an Initial Value Problem (IVP) with non-homogeneous ODE, you would have to, on top of everything already done, use the given initial conditions to find $c_1$ and $c_2$ (by solving a linear system). It works exactly the same as in the examples done in last recitation.

**Example 31**

Determine the general solution of the non-homogeneous linear differential equation $y'' - 4y' - 32y = 6e^{-3t}$.

---

1. Finding $y_h$: consider first the homogeneous equation $y'' - 4y' - 32y = 0$. Its characteristic equation is $r^2 - 4r - 32 = 0$, which becomes $(r-8)(r+4) = 0$. This implies, as usual, that we have $y_h = c_1 e^{8t} + c_2 e^{-4t}$, with $c_1, c_2 \in \mathbb{R}$.

2. Finding $y_p$: as the non-homogeneous term in the original equation is an exponential, we try $y_p = Ae^{-3t}$, and substitute it together with its derivatives $y_p' = -3Ae^{-3t}$ and $y_p'' = 9Ae^{-3t}$ into the original equation to find the value of $A$ which makes $y_p$ into a solution. We have that
$$9Ae^{-3t} - 4(-3Ae^{-3t}) - 32Ae^{-3t} = 6e^{-3t} \implies 9A + 12A - 32A = 6,$$
so $A = -6/11$, since $e^{-3t}$ may be cancelled everywhere. Therefore, the particular solution is $y_p = (-6/11)e^{-3t}$.

3. Putting everything together: the general solution of the original non-homogeneous equation is
$$y = -\frac{6}{11}e^{-3t} + c_1 e^{8t} + c_2 e^{-4t},$$
with $c_1, c_2 \in \mathbb{R}$.

**Example 32**

Determine the general solution of the following non-homogeneous linear differential equations:

(a) $y'' - y = 3\sin(2t)$.

(b) $y'' + y = 3\cos t$.

---

(a) Again, we organize ourselves with three steps:

1. Finding $y_h$: we start considering the associated homogeneous equation $y'' - y = 0$, whose characteristic equation $r^2 - 1 = 0$ has roots 1 and $-1$, to obtain that its general solution is $y_h = c_1 e^t + c_2 e^{-t}$, with $c_1, c_2 \in \mathbb{R}$.
2. Finding $y_p$: as the non-homogeneous term is trigonometric, we may try $y_p = A\sin(2t) + B\cos(2t)$. Of course one might expect that $B = 0$ as the right side of the equation doesn't have any $y'$ term, but for more complicated equations it is safer to keep both the sin and cos terms, as the derivative of each of them equals the other, and things balance themselves out (this is not a formal argument, but just a rough intuition). Anyway, we have that $y_p'' = -4A\sin(2t) - 4A\cos(2t)$, and thus
$$y_p'' - y_p = 3\sin(2t) \implies -5A\sin(2t) - 5B\cos(2t) = 3\sin(2t),$$
so that $A = -3/5$ and $B = 0$ (as expected).
3. Putting everything together: the solution of the original non-homogeneous equation is
$$y = -\frac{3}{5}\sin(2t) + c_1 e^t + c_2 e^{-t},$$
with $c_1, c_2 \in \mathbb{R}$.

(b) This time, we have:

1. Finding $y_h$: we consider $y'' + y = 0$ instead, whose characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$. This says that $e^{it}$ is a complex solution, but we would like to have two linearly independent real solutions instead. They are obtained by using Euler's formula to write $e^{it} = \cos t + i\sin t$, and using that the real and imaginary parts will be real solutions. So, we have $y_h = c_1\cos t + c_2\sin t$, with $c_1, c_2 \in \mathbb{R}$.
2. Finding $y_p$: the obvious guess is $y_p = A\sin t + B\cos t$. Plugging this into the original non-homogeneous equation leads to a disaster: $0 = 3\cos t$, which is complete nonsense.

   This disaster, however, is known as **resonance**. Namely, the reason why the initial guess has failed is because the right hand side (which in this case is $3\cos t$) was already a solution of the associated homogeneous equation (choose $c_1 = 0$ and $c_2 = 3$ in the formula for $y_h$). In vague terms, this creates an "artificial double-root effect" on the homogeneous equation, so in the same way that when $r$ was a double root of the characteristic equation $ar^2 + br + c = 0$ we had to, in addition to the solution $e^{rt}$, consider a second solution $te^{rt}$, we multiply our old guess for $y_p$ by $t$ and try again.

   Take two: let $y_p = At\sin t + Bt\cos t$. Plugging this into the original non-homogeneous equation leads (after 30 seconds or so of calculations) to
$$2A\cos t - 2B\sin t = 3\cos t \implies A = \frac{3}{2} \quad \text{and} \quad B = 0.$$

It was to be expected that terms with $t\cos t$ and $t\sin t$ should completely dissapear: the introduction of the extra factors of $t$ was just meant to kill the resonance, but the original right side did not have any such terms. Hence, $y_p = \frac{3t}{2}\sin t$.

3. Putting everything together: the general solution of the original non-homogeneous equation is

$$y = \frac{3t}{2}\sin t + c_1\cos t + c_2\sin t,$$

with $c_1, c_2 \in \mathbb{R}$.

**Remark** (Nightmare fuel)**.** It's possible for resonance to persist even after multiplying our initial guess by $t$. Imagine that instead of $y'' + y$, we had something like scary like

$$y^{(4)} + 2y'' + y = 3\cos t,$$

such that $3\cos t$ and $3t\cos t$ are both solutions of the associated homogeneous equation (verify it yourself, or try to understand how I reverse engineered it). The point is that the guess $At\sin t + Bt\sin t$ would also fail! Since crying is not an option, we go one step further and try $y_p = At^2\sin t + Bt^2\cos t$ instead. Exercise for the masochists: check that this works. So, what's the moral of the story? Keep multiplying your guesses by $t$ until the resonance goes away. It is probably a terrible idea to try to identify immediately whether the given equation has resonance, and try to find the smallest power of $t$ you need to use to kill it. Do not let this cause you any paranoia: try what you want to try, and if it doesn't work, just multiply by $t$.

Let's conclude with one final practice for guesses:

**Example 33**

What are the correct trial solutions for the following non-homogeneous linear differential equations? Is there resonance happening? Do **not** solve the equations.

(a) $y'' + 2y' + 2y = 5e^{-2t}\cos t$.

(b) $y'' + y' - y = 3t^4 - 3t^3 + t$.

(c) $y'' - y = 25te^{-t}\sin(3t)$.

(d) $y^{(3)} - 3y'' + 3y' - y = 2022e^t$.

---

(a) Here, we have that $5e^{-2t}\cos t$ has the form "polynomial of degree 0" times "exponential" times "trigonometric function". So the first guess ought to be

$$y_p = Ae^{-2t}\cos t + Be^{-2t}\sin t.$$

The paranoia created by the previous exercise now dictates that we should ask ourselves whether resonance happens. The trick here is to think of the term $e^{-2t}\cos t$ as a single block, coming from the complex pair $r = -2 \pm i$. Is $-2+i$ a root of the associated characteristic equation $r^2 + 2r + 2 = 0$? If yes, there is resonance. If not, there is no resonance. In this case, there is no resonance.

(b) Since $3t^4 - 3t^3 + t$ is a polynomial of degree 4, the guess is that the trial solution should also be a polynomial of degree 4, namely,

$$y_p = At^4 + Bt^3 + Ct^2 + Dt + E.$$

If resonance were to happen, part of this trial solution should be a solution of the associated homogeneous equation. The only terms under this risk are $Dt$ and $E$, in case 0 were a root or double root of the characteristic equation $r^2 + r - 1 = 0$. Since it is not, there is no resonance.

(c) This is similar to item (a), in that $25te^{-t}\sin(3t)$ has the form "polynomial of degree 1" times "exponential" times "trigonometric function". So the first guess ought to be

$$y_p = (At + B)e^{-t}\cos(3t) + (Ct + D)e^{-t}\sin(3t).$$

We can think of the terms $e^{-t}\cos(3t)$ and $e^{-t}\sin(3t)$ as coming from the complex pair $-1 \pm 3i$. Since $-1 + 3i$ is not a root of the associated characteristic equation $r^2 - 1 = 0$, there is no resonance.

(d) The first guess here is $y_p = Ae^t$. The exponential $e^t$ comes from the characteristic root $r = 1$. Is $r = 1$ a root of the characteristic equation $r^3 - 3r^2 + 3r - 1 = 0$? Yes, there is resonance! In fact, it is a **triple** root (as the equation factors as $(r-1)^3 = 0$), so we multiply the initial guess by $t^3$ to obtain the correct guess $y_p = At^3e^t$.

# 12 November 7th

## 12.1 Separation of variables and boundary value problems

Back in August 22$^{nd}$, we introduced the **Heat Flow Problem** as an attempt to motivate everything else that was going to happen in this class. It is a particular case of a **Partial Differential Equation** ("PDE", for short). They are considerably harder to deal with than everything we have seen so far. The time has come to solve it.

Recall the setup:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \beta \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

- Given: a wire of length $L$ with diffusivity constant $\beta$ and initial temperature distribution $f(x)$;
- Assumption: the temperature of the wire at its endpoints – located at $x = 0$ and $x = L$ – will remain fixed (and equal to, say, $0^\circ$C);
- Goal: predict future temperature distributions.

The strategy for solving such a starts with a simple technique, called **separation of variables**. Namely, we write $u(x,t) = X(x)T(t)$ for some single-variable functions $X$ and $T$, and see what the PDE says about $X$ and $T$. More precisely, we want to replace the PDE for $u$ with ODEs for $X$ and $T$. We will illustrate the procedure with a concrete heat flow problem.

**Example 34**

*Solve the following heat flow problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 3\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin x - 6\sin(4x), & 0 < x < \pi \end{cases}$$

Let's try to organize everything in steps.

- **Step 1:** We start making a separation of variables, $u(x,t) = X(x)T(t)$. The PDE itself becomes $X(x)T'(t) = 3X''(x)T(t)$. We then have that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{3T(t)} = -\lambda \in \mathbb{R},$$

as the left side doesn't depend on $t$ and the right side doesn't depend on $x$. The reason for the negative sign on $\lambda$ is a matter of convenience (as to make plus signs appear next and create "positive eigenvalues" for the boundary value problem involving $X$ in the next steps).

- **Step 2:** Reorganize the ODEs and boundary conditions. We have the equations
$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T'(t) + 3\lambda T(t) = 0,$$
and we'll use the first one to find out which values of $\lambda$ may actually occur when looking for nontrivial solutions. To do so, we also need to see what the initial condition $u(0,t) = u(\pi,t) = 0$ means in terms of $X$ and $T$. But this is easy: it becomes just the initial condition $X(0) = X(\pi) = 0$ (or else $T(t) = 0$ and thus $u(x,t) = 0$, which makes no physical sense).

- **Step 3:** Solve the *boundary value problem*
$$(\text{BVP}) \begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(\pi) = 0 \end{cases}$$
Unlike IVPs, BVPs in general do not admit nontrivial solutions. The question then becomes: for which values of $\lambda$ does the BVP admit nontrivial solutions (that is, we are *not* forced onto $c_1 = c_2 = 0$). Such values for $\lambda$ are called the *eigenvalues* of the BVP. We analyze three cases for $\lambda$.
  - If $\lambda < 0$, then $X(x) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{\lambda}t}$, so $X(0) = X(\pi) = 0$ gives us that $c_1 = c_2 = 0$, and so $X = 0$, which is no good.
  - If $\lambda = 0$, then we have $X(x) = c_1 + c_2 x$, so $X(0) = X(\pi) = 0$ gives us that $c_1 = c_2 = 0$, and so $X = 0$, which is again no good.
  - If $\lambda > 0$, then $X(x) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t)$. Now $X(0) = 0$ means that $c_1 = 0$, while $X(\pi) = 0$ says that $\sin(\pi\sqrt{\lambda}) = 0$ (provided $c_2 \neq 0$). This can only occur if $\pi\sqrt{\lambda} = n\pi$ for some integer $n$, meaning that we must have that $\lambda = n^2$.

  So, for each integer $n$, we have found a solution $X_n(x) = a_n \sin(nx)$, where $a_n$ is a real number (we will keep relabeling constants as needed). We call $X_n$ an *eigenfunction* of the BVP.

- **Step 4:** Find the corresponding $T_n(t)$. Solving the first order differential equation $T_n'(t) = -3n^2 T_n(t)$ leads to $T_n(t) = b_n e^{-3n^2 t}$ for some second real constant $b_n$.

- **Step 5:** Setting $c_n = a_n b_n$, put $X_n$ and $T_n$ together into
$$u_n(x,t) = X_n(t)T_n(t) = c_n e^{-3n^2 t} \sin(nx).$$

For each $n$, this function $u_n$ satisfies both the PDE and the boundary value conditions, but in general it does not satisfy the initial distribution condition $u(x,0) = \sin x - 6\sin(4x)$. To achieve this, we consider the series $u = \sum_{n\geq 1} u_n$, that is, $u(x,t) = \sum_{n\geq 1} c_n \mathrm{e}^{-3n^2 t}\sin(nx)$. As the PDE itself and the endpoint conditions are linear and homogeneous, any linear combination – finite or infinite – of $u_n$'s will also result in a solution. Therefore...

- **Step 6:** ... we must find the $c_n$'s for which the initial distribution condition also works. As

$$\begin{aligned} u(x,0) &= \sum_{n\geq 1} c_n \mathrm{e}^{-3n^2 t}\sin(nx)\Big|_{t=0} \\ &= \sum_{n\geq 1} c_n \sin(nx) \\ &= c_1\sin x + c_2\sin(2x) + c_3\sin(3x) + c_4\sin(4x) + \cdots \\ &= \sin x - 6\sin(4x), \end{aligned}$$

we must have that $c_1 = 1$ and $c_4 = -6$, with $c_n = 0$ for every $n$ not equal to 1 or 4.

- **Step 7:** Write the final solution: $u(x,t) = \mathrm{e}^{-3t}\sin x - 6\mathrm{e}^{-48t}\sin(4x)$.

**Remark.** Boundary value problems as the one in **Step 3** only admit positive eigenvalues. In other words, when looking at possibilities for $\lambda$ we will always have that $\lambda \leq 0$ leads to $X = 0$. When solving concrete problems, one could directly jump to the case where $\lambda > 0$, but it is still instructive to eliminate $\lambda \leq 0$ to keep track of what's going on in each step of the solution.

Solving the Heat Flow Problem abstractly, carrying the abstract parameters $\beta$ and $L$, one arrives at what we'll call a **prototype solution to the Heat Flow Problem**:

$$u(x,t) = \sum_{n\geq 1} c_n \mathrm{e}^{-\beta(n\pi/L)^2 t}\sin\left(\frac{n\pi x}{L}\right).$$

You have a choice to make now: remember all the several little steps described in the example above, or memorize this very unpleasant formula.

If taking this formula for granted, the example above would start in **Step 6**. However, if $f(x)$ is not a linear combination of sines, finding the $c_n$'s may be tricky. Here's where Fourier series come in: almost all functions $f(x)$ may be expressed as an infinite linear combination of sines and, once this is done, reading the coefficients $c_n$ becomes much easier. More precisely, a function $f(x)$ on an interval $0 < x < L$ may be written as

$$f(x) = \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients $b_n$ are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Solving such integrals almost always requires **integration by parts**. In this case, it will turn out that

$$u(x,0) = f(x) \implies \sum_{n\geq 1} c_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{L}\right) \implies c_n = b_n.$$

Lastly, we note that the same strategy works to solve, for example, a **vibrating string problem**:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L, \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), & 0 < x < L, \end{cases}$$

**Steps 1 through 3** still work with **no changes**. The first difference is in **Step 4**, as now the ODE for $T_n(t)$ has order 2 instead of 1, and thus there are two sequences of coefficients to solve for (as opposed to just $c_n$'s), and those are found by looking at the Fourier series of both $f(x)$ and $g(x)$.

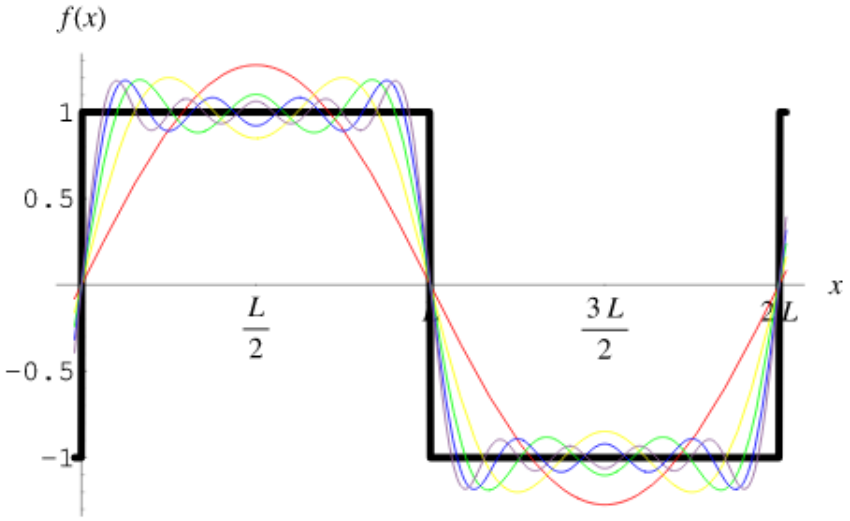
# 13 November 14th

## 13.1 Fourier Series

We finally explore a bit of Fourier series here. To get some first intuition, let's compare it with a type of series we have already studied before:

- **Taylor series:** its partial sums (Taylor polynomials) approximate a function, near a given point, with higher and higher degree polynomial expressions.
- **Fourier series:** its partial sums (trigonometric polynomials) approximate a function by superposing more and more "waves" of various frequencies.

Imagine something like the following picture[4]:




In the above picture, the indicated waves correspond to partial sums of a Fourier series. But how to actually compute them? The Fourier expansion[5] of a function $f(x)$ on a symmetric interval $[-L, L]$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right), \tag{13.1}$$

where

$$\text{i) } a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad \text{and} \quad \text{ii) } b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{13.2}$$

Here, the formula for $a_n$ holds for $n = 0, 1, 2, \ldots$, while the formula for $b_n$ holds for $n = 1, 2, \ldots$. There are many things to unpack here. First, there's no $b_0$ terms because trying to plug $n = 0$ into the formula for $b_n$ gives just 0. Second, the coefficient $a_0$

[4] Taken from https://mathworld.wolfram.com/FourierSeriesSquareWave.html.

[5] "Fourier series" and "Fourier expansion" are used interchangeably. "Fourier transform" is something completely different, though.

plays a different role than the coefficients $a_n$ for $n \geq 1$, and must always be addressed separately. One reason for this is that their "qualitative" behavior is very distinct. For example, the constant function 1 regarded as a wave has no amplitude, and its usual antiderivative, $x$, is a polynomial, while $\cos(n\pi x/L)$ has varying amplitude and its usual antiderivative is trigonometric. The factor of $1/2$ in (13.1) to allow us to include $n = 0$ in (13.2-i). Lastly, when expressing the Fourier series of a function, one uses the approximate sign $\sim$ instead of an equality sign, as a reminder that convergence issues of Fourier series are more subtle than convergence issues for, say, Taylor series.

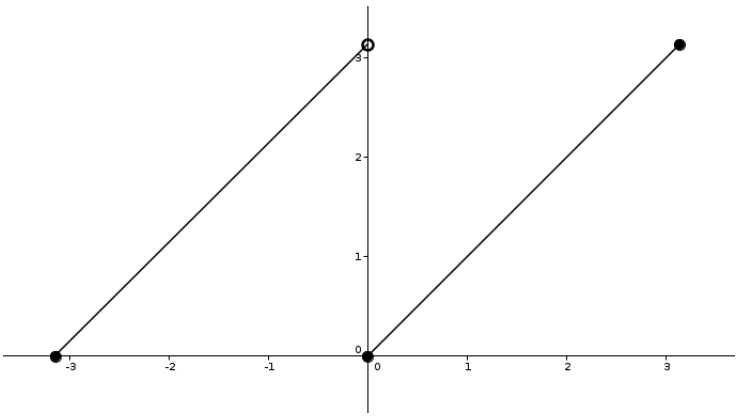
It's also convenient to note that while the argument $n\pi x/L$ doesn't look exactly friendly, in most problems we have to deal with, the number $L$ will be an integer multiple of $\pi$, which makes things more tractable.

**Example 35**

*Compute the Fourier expansion of the piecewise function $f$ given by*

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \pi, \\ x + \pi, & \text{if } -\pi \leq x < 0. \end{cases}$$

We may start trying to get some intution for what the graph of this function looks like. See the next figure.




We must simply compute $a_0$, $a_n$, and $b_n$, and insert the results in a series. From the figure, we immediately have that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\cdot\frac{\pi^2}{2}\cdot 2 = \pi,$$

as we know that integrals of positive functions compute areas under graphs, and we have two triangles with both base and height equal to $\pi$. As for $a_n$ and $b_n$ with $n \geq 1$, we must break the integral from $-\pi$ to $\pi$ into two integrals, so we can actually use the concrete expressions for $f(x)$ (on each subinterval) given to us.

For example:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} (x+\pi)\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} x\cos(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\cos(nx)\,\mathrm{d}x + \cancel{\int_{-\pi}^{\pi} x\cos(nx)\,\mathrm{d}x}\right)_{0} \\
&= \int_{-\pi}^{0} \cos(nx)\,\mathrm{d}x \\
&= \frac{\sin(nx)}{n}\bigg|_{-\pi}^{0} \\
&= 0.
\end{aligned}$$

Again, breaking the original integral into two was needed so we could use the concrete expressions given for $f(x)$ on each interval. Being able to join things back together on the $\int_{-\pi}^{\pi} x\cos(nx)\,\mathrm{d}x$ was a convenient coincidence due to the fact that $x$ appeared in both expressions defining $f(x)$. This integral is zero for symmetry reasons: **the integral of an odd function over a symmetric interval vanishes** (namely, $x$ is odd and $\cos(nx)$ is even, so the product $x\cos(nx)$ is odd). Dealing with $b_n$'s is similar, this time using that **the integral of an even function over a symmetric interval equals twice the integral over the right (or left) half of the interval**. We have that:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} (x+\pi)\sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} x\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\sin(nx)\,\mathrm{d}x + \int_{-\pi}^{\pi} x\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\sin(nx)\,\mathrm{d}x + 2\int_{0}^{\pi} x\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(nx)\bigg|_{-\pi}^{0} + 2\left(-\frac{x}{n}\cos(nx)\bigg|_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos(nx)\,\mathrm{d}x\right)\right) \\
&= \frac{1}{\pi}\left(-\frac{\pi}{n}(1-(-1)^n) + 2\left(-\frac{\pi}{n}(-1)^n + \cancel{\frac{\sin(nx)}{n^2}\bigg|_{0}^{\pi}}\right)_{0}\right)
\end{aligned}$$

$$= -\frac{1}{n}(1-(-1)^n + 2(-1)^n)$$
$$= -\frac{(1+(-1)^n)}{n}$$

by using that $\cos(n\pi) = (-1)^n$ for any integer $n$. In green, we have also used integration by parts with

$$u = x \qquad \mathrm{d}v = \sin(nx)\,\mathrm{d}x$$
$$v = -\frac{1}{n}\cos(nx) \qquad \mathrm{d}u = \mathrm{d}x$$

In any case, we have obtained the Fourier expansion

$$f(x) \sim \frac{\pi}{2} - \sum_{n\geq 1} \frac{1+(-1)^n}{n}\sin(nx) \quad \text{on } [-\pi,\pi],$$

as desired. Since $1+(-1)^n$ equals 0 when $n$ is odd and 2 when $n$ is even, we may set $n = 2k$ and rewrite our answer as

$$f(x) \sim \frac{\pi}{2} - \sum_{k\geq 1} \frac{1}{k}\sin(2kx) \quad \text{on } [-\pi,\pi],$$

after simplifying $2/(2k) = 1/k$.

In the above example, we were able to find the Fourier expansion of a function $f(x)$ which was not continuous (namely, the figure shows a jump discontinuity at $x = 0$). This is another important difference between Taylor series and Fourier series. For Taylor series, the function must have all derivatives existing at the chosen center point, while a Fourier series does not require the choice of a center point (although one could reasonably argue that the center in this case is 0) or derivatives to exist. The only thing we must be able to do is to compute the relevant integrals, but integrals are insentitive to a countable number of discontinuities.

# 14 November 21st

## 14.1 Half-interval Fourier series

So far, given a Heat Flow Problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \beta \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L, \end{cases}$$

we have obtained the prototype solution

$$u(x,t) = \sum_{n \geq 1} c_n \mathrm{e}^{-\beta(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right),$$

which verified both the heat equation itself and the boundary conditions, but not necessarily the initial temperature distribution condition $u(x,0) = f(x)$. Then, finding the $c_n$'s for this last condition to be satisfied was easy when $f(x)$ was a linear combination of sine functions, but not otherwise. We used this as a motivation to start studying Fourier series. Namely, for a function $f(x)$ defined on a **symmetric interval** $[-L, L]$, we had

$$f(x) \sim \frac{a_0}{2} + \sum_{n \geq 1} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

where

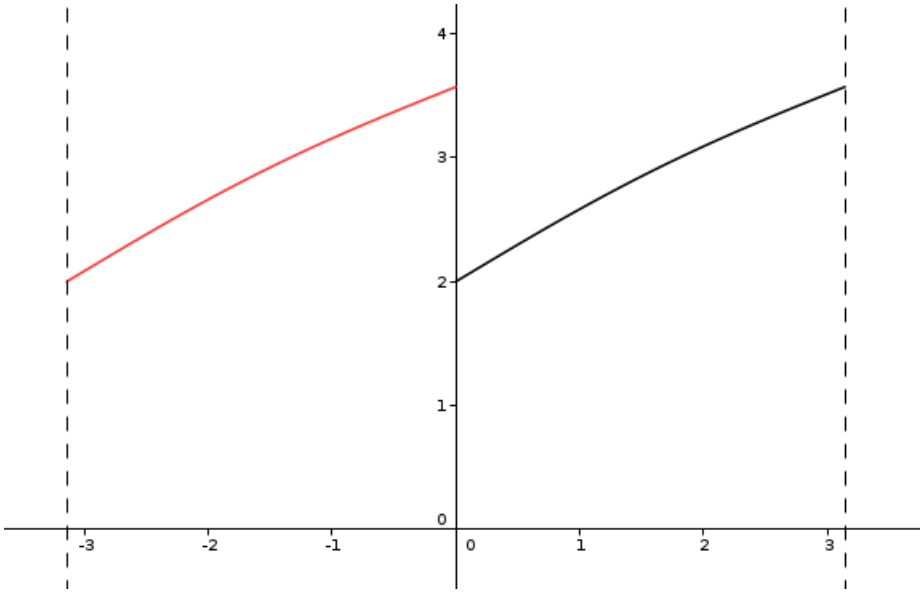
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

There are two issues here: first, the heat flow problem has initial temperature distributions defined only on $[0, L]$ as opposed to $[-L, L]$ and, secondly, a general Fourier series has terms involving cosines, which are not to be expected when solving a Heat Flow Problem. Something does not add up, and we need to make sense of it.

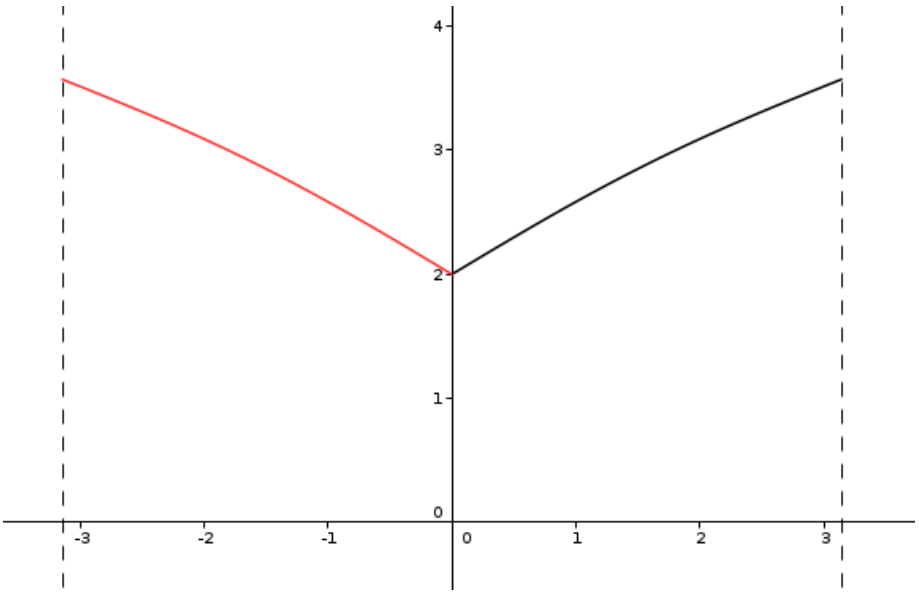
Our starting point is: how to make sense of a Fourier series for a function defined only on $[0, L]$? Note that asking yourself whether a function defined on a half-interval $[0, L]$ is even or odd makes no sense: it would have to be defined on a symmetric interval $[-L, L]$ to begin with.

The idea here is to start with a function $f(x)$ defined on $[0, L]$, extend it to a function $\widetilde{f(x)}$ on $[-L, L]$, compute the Fourier series of $\widetilde{f(x)}$, and then restrict it back to the original interval $[0, L]$. However, there is more than one way to extend $f(x)$. We decide which extension to use depending on the problem we are trying to solve.

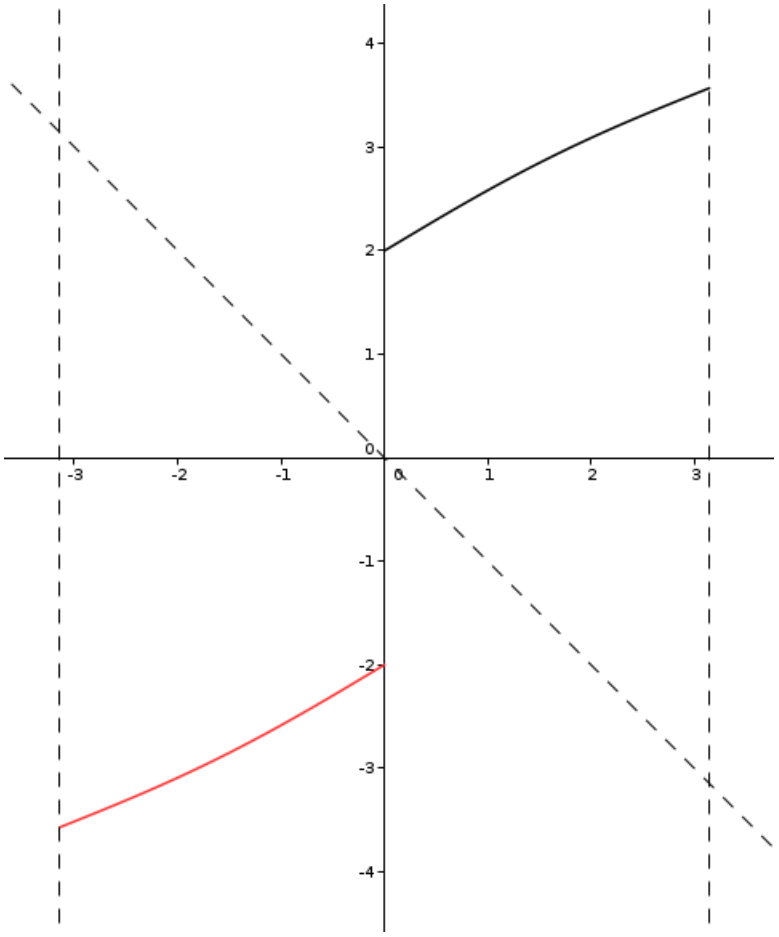
- **Periodic extension.** Copy and paste the function defined on $[0, L]$ to the interval $[-L, 0]$:




- **Even extension.** Flip it across the $y$-axis:



- **Odd extension.** Reflect the function across the origin:




Recall that if $\widetilde{f(x)}$ is even, then $b_n = 0$ for all $n$, since the full Fourier series of an even function should not have odd/sin terms. Similarly, if $\widetilde{f(x)}$ is odd, then $a_n = 0$ for all $n$, since the full Fourier series of an odd function should not have even/cos terms. In other words, symmetries of a function reflect into symmetries of its Fourier series.

What would be the Fourier series for the even extension $\widetilde{f(x)}$ of $f(x)$? Compute

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} \widetilde{f(x)} \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\int_{0}^{L} \widetilde{f(x)} \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\ &= \frac{2}{L}\int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \end{aligned}$$

since $\widetilde{f(x)} = f(x)$ on $[0, L]$. A similar calculation gives us an updated formula for $b_n$ in the odd-extension case. We summarize below our conclusions:

- **Sine series:** This is done for a function $f(x)$ defined on a **half-interval** $[0, L]$, and we have

$$f(x) \sim \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad n = 1, 2, \ldots$$

This is important when dealing with heat flow problems, as the initial temperature distribution is defined on an interval $[0, L]$, and not $[-L, L]$.

- **Cosine series:** This is done for a function $f(x)$ defined on a **half-interval** $[0, L]$, and we have

$$f(x) \sim \frac{a_0}{2} + \sim \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad n = 0, 1, 2, \ldots$$

This sort of expansion is useful for solving problems similar to heat flow problems, but on which the boundary condition $u(0,t) = u(L,t) = 0$ is replaced[6] with $(\partial u/\partial x)(0,t) = (\partial u/\partial x)(L,t) = 0$ – the resulting prototype solution has cosines instead of sines, and the rest is history.

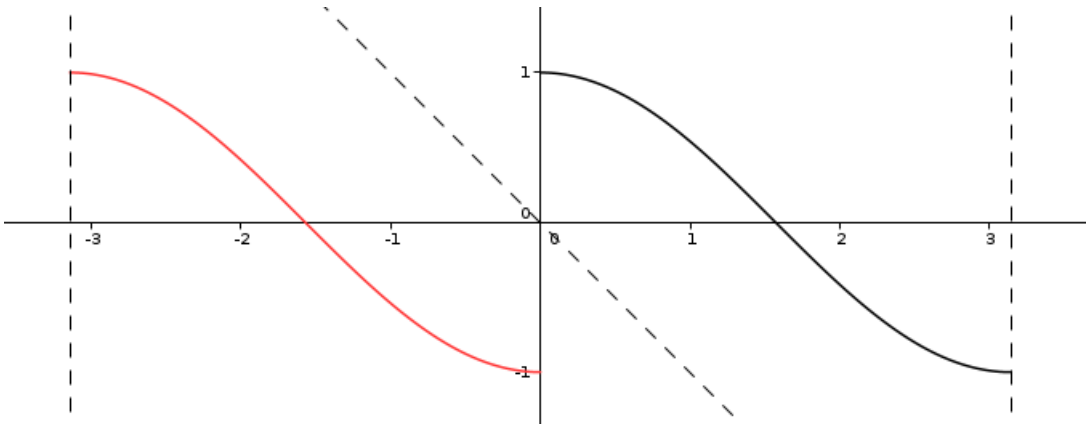
There is no way to simplify the (already adjusted!) formulas for sine series and cosine series. With this in place, let's see what happens with a concrete example:

**Example 36**

*Determine the Fourier sine-expansion of* $\cos x$ *on* $[0, \pi]$.

---

Let's start by understanding what is the odd extension $\widetilde{\cos x}$ of $\cos x$ looks like (and noting that the fact that $\cos x$, **when considered on the full interval** $[-\pi, \pi]$ **to begin with**, was already even, is irrelevant):




In this particular case, the periodic extension and the odd extension agree. We already know that $a_n = 0$ for all $n$ by symmetry reasons. As for the coefficients $b_n$,

[6] **Exercise:** what are the eigenvalues $\lambda$ and eigenfunctions $X$ to the equation $X''(x) + \lambda X(x) = 0$ subject to $X'(0) = X'(L) = 0$ instead of $X(0) = X(\pi) = 0$?

we have that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \widetilde{\cos x}\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} \widetilde{\cos x}\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} \cos x\sin(nx)\,\mathrm{d}x,$$

since $\widetilde{\cos x}\sin(nx)$ is even (as the product of the two odd functions $\widetilde{\cos x}$ and $\sin(nx)$), and $\widetilde{\cos x} = \cos x$ for $x$ in the right interval $[0,\pi]$ (this is what it means to say that $\widetilde{\cos x}$ is an **extension** of $\cos x$). Of course, when solving problems like this, you don't have to go and write the above step carrying the extension of the given function: go ahead and compute $b_n$ (or $a_n$, for even extensions) with the extra coefficient of 2 and the integral being carried only over the "right" interval. The point of doing this step here is illustrating that while the textbook [1] presents different formulas for Fourier expansions, Fourier sine-expansions, and Fourier-cosine expansion, they're really the same thing, and no extra effort on memorizing things should be made.

In any case, it remains to compute this integral. For that, we must rely on product-to-sum trigonometric identities:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \cos x\sin(nx)\,\mathrm{d}x\\
&= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\sin((n+1)x) + \sin((n-1)x))\,\mathrm{d}x\\
&= \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) + \sin((n-1)x)\,\mathrm{d}x\\
&= -\frac{1}{\pi}\left(\frac{\cos((n+1)x)}{n+1} + \frac{\cos((n-1)x)}{n-1}\right)\Bigg|_0^{\pi}\\
&= -\frac{1}{\pi}\left(\frac{(-1)^{n+1}-1}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right)\\
&\overset{(*)}{=} \frac{1+(-1)^n}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)\\
&= \frac{2n(1+(-1)^n)}{\pi(n^2-1)},
\end{aligned}$$

where in $(*)$ we have used that $(-1)^{n+1} = (-1)^{n-1}$ (as the powers differ by an even number) and distributed the negative sign to carry the simplification. This means that we have

$$\cos x \sim \sum_{n\geq 1} \frac{2n(1+(-1)^n)}{\pi(n^2-1)}\sin(nx) \quad \text{on } [0,\pi].$$

And here is how to apply these ideas to solve a concrete Heat Flow Problem:

**Example 37**

*Solve the following heat flow problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 10\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & \\ u(x,0) = \cos x, & 0 < x < \pi \end{cases}$$

---

We may take

$$u(x,t) = \sum_{n\geq 1} c_n \mathrm{e}^{-10n^2 t}\sin(nx)$$

as our starting point, and find the $c_n$'s that work. As

$$\sum_{n\geq 1} c_n \mathrm{e}^{-10n^2 t}\sin(nx)\bigg|_{t=0} = \sum_{n\geq 1} c_n \sin(nx) = \cos x.$$

By previous calculations, we already have that

$$\cos x \sim \sum_{n\geq 1} \frac{2n(1+(-1)^n)}{\pi(n^2-1)}\sin(nx) \quad \text{on } [0,\pi],$$

and so

$$c_n = \frac{2(1+(-1)^n)n}{\pi(n^2-1)}$$

for every $n \geq 1$. The final solution to the given heat flow problem is

$$u(x,t) = \sum_{n\geq 1} \frac{2n(1+(-1)^n)}{\pi(n^2-1)}\mathrm{e}^{-10n^2 t}\sin(nx).$$

Let's also see one example of a cosine series:

**Example 38**

*Determine the Fourier cosine-expansion of* $\mathrm{e}^{-x}$ *on* $[0,\pi]$.

---

This time, there are only $a_n$'s to compute. Namely, we have that

$$a_n = \frac{2}{\pi}\int_0^\pi \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x.$$

We may allow $n = 0$ and compute $a_0$ together with the other $a_n$'s, provided that at the end we remember to divide $a_0$ by 2 when plugging its value in the series. To avoid clutter, let's first find an anti-derivative for $\mathrm{e}^{-x}\cos(nx)$, using integration

by parts. We start by letting $u = \mathrm{e}^{-x}$ and $\mathrm{d}v = \cos(nx)\,\mathrm{d}x$, so that $\mathrm{d}u = -\mathrm{e}^{-x}\,\mathrm{d}x$ and $v = \sin(nx)/n$ lead to

$$\int \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x = \frac{1}{n}\,\mathrm{e}^{-x}\sin(nx) + \frac{1}{n}\int \mathrm{e}^{-x}\sin(nx)\,\mathrm{d}x.$$

Now, we apply integration by parts again in the very last integral, with $u = \mathrm{e}^{-x}$ and $\mathrm{d}v = \sin(nx)\,\mathrm{d}x$, so $\mathrm{d}u = -\mathrm{e}^{-x}\,\mathrm{d}x$ and $v = -\cos(nx)/n$. Here, this is the only correct choice to make, as switching the roles of $u$ and $v$ would **undo** the previous integration by parts. We obtain

$$\int \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x = \frac{1}{n}\,\mathrm{e}^{-x}\sin(nx) + \frac{1}{n}\left(-\frac{1}{n}\,\mathrm{e}^{-x}\cos(nx) - \frac{1}{n}\int \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x\right).$$

Instead of applying integration by parts a third time, we now solve for the entire integral $\int \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x$ in one shot. Namely,

$$\int \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x = \frac{1}{n}\,\mathrm{e}^{-x}\sin(nx) - \frac{1}{n^2}\,\mathrm{e}^{-x}\cos(nx) - \frac{1}{n^2}\int \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x$$

implies that

$$\left(1 + \frac{1}{n^2}\right)\int \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x = \frac{1}{n}\,\mathrm{e}^{-x}\sin(nx) - \frac{1}{n^2}\,\mathrm{e}^{-x}\cos(nx),$$

which becomes

$$\int \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x = \frac{1}{n^2+1}\left(n\mathrm{e}^{-x}\sin(nx) - \mathrm{e}^{-x}\cos(nx)\right).$$

Back to computing $a_n$'s. We directly have that

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \mathrm{e}^{-x}\cos(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\frac{1}{n^2+1}\left(\cancelto{0}{n\mathrm{e}^{-x}\sin(nx)} - \mathrm{e}^{-x}\cos(nx)\right)\Bigg|_0^\pi \\ &= \frac{2}{\pi(n^2+1)}(-\mathrm{e}^{-\pi}(-1)^n + 1). \end{aligned}$$

Note that $a_0 = \dfrac{2}{\pi}(1 - \mathrm{e}^{-\pi})$. The Fourier cosine series we're looking for is

$$\mathrm{e}^{-x} \sim \frac{1}{\pi}(1 - \mathrm{e}^{-\pi}) + \sum_{n\geq 1}\frac{2(1-(-1)^n\mathrm{e}^{-\pi})}{\pi(n^2+1)}\cos(nx).$$

# 15 November 28th

## 15.1 More examples

The goal today is just to explore and practice with further examples.

The first example below had been assigned as a homework problem last year (but not this year[7]) – it does not involve complicated integrations, but requires a solid understanding of how to deal with symmetries and piecewise functions.

**Example 39**

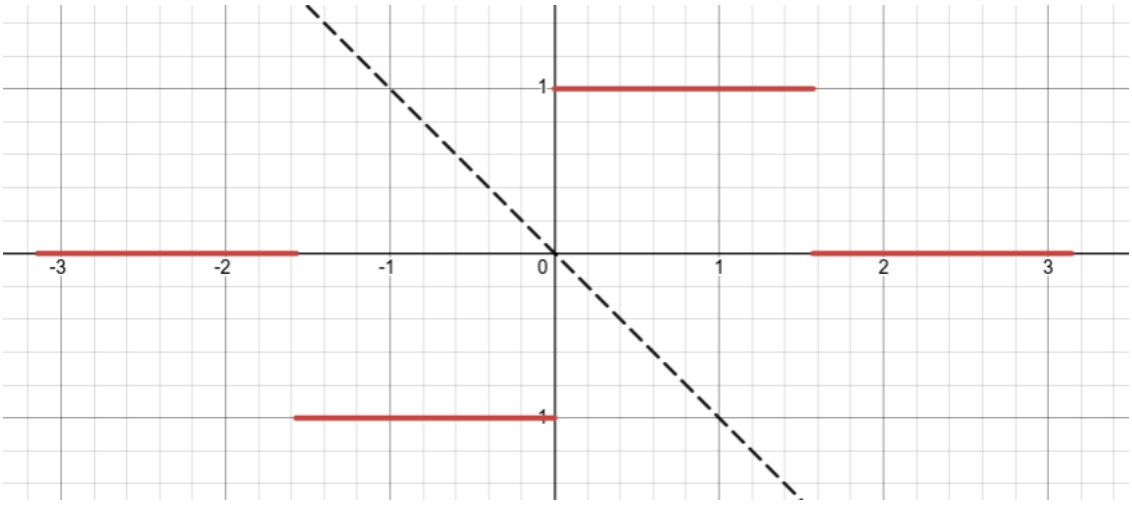
*Compute the Fourier series of*

$$f(x) = \begin{cases} 0, & \text{if } -\pi \leq x < -\pi/2 \\ -1, & \text{if } -\pi/2 \leq x < 0 \\ 1, & \text{if } 0 \leq x < \pi/2, \\ 0, & \text{if } \pi/2 \leq x \leq \pi \end{cases}$$

*on the interval* $[-\pi, \pi]$.

---

The first thing to observe here is that the given function $f$ is an odd function. See its graph below to recognize symmetry about the origin:




As a consequence, $f(x)\cos(nx)$ is an odd function as well, even when $n = 0$, so that $a_n = 0$ for all $n \geq 0$. It remains to compute the coefficients $b_n$. As the product

[7] For reasons I don't understand.

$f(x)\sin(nx)$ is an even function, we have that

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi/2} f(x)\sin(nx)\,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi/2} 1\sin(nx)\,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} 0\sin(nx)\,\mathrm{d}x \to 0 \\
&= \frac{2}{\pi}\int_{0}^{\pi/2} \sin(nx)\,\mathrm{d}x \\
&= \frac{-2}{n\pi}\cos(nx)\Big|_{0}^{\pi/2} \\
&= \frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right).
\end{aligned}$$

Therefore, the resulting Fourier series reads

$$f(x) \sim \sum_{n\geq 1}\frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin(nx).$$

**Optional:** As $n$ ranges over the integers, $\cos(n\pi/2)$ takes all three values $1, -1, 0$. Unlike what happened with $\cos(n\pi) = (-1)^n$, there isn't one single formula for $\cos(n\pi/2)$ in terms of powers of $-1$. The best we can do is to write

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ (-1)^k, & \text{if } n = 2k \text{ is even.} \end{cases}$$

It is probably not a good idea to try to use this to simplify things.

**Example 40**

*Solve the following heat flow problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 3\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & \\ u(x,0) = x^2, & 0 < x < \pi \end{cases}$$

We may take

$$u(x,t) = \sum_{n\geq 1} c_n \mathrm{e}^{-3n^2 t}\sin(nx)$$

as our starting point, and find the $c_n$'s that work. As

$$\sum_{n\geq 1} c_n \mathrm{e}^{-3n^2 t}\sin(nx)\bigg|_{t=0} = \sum_{n\geq 1} c_n \sin(nx) = x^2,$$

we see that the Fourier *sine* series of $f(x) = x^2$ on $[0,\pi]$ is needed to continue. For half-interval expansions, the adjusted formula for $b_n$ becomes

$$b_n = \frac{2}{\pi}\int_0^\pi x^2\sin(nx)\,\mathrm{d}x.$$

To avoid clutter, we first find an antiderivative of $x^2\sin(nx)$, using integration by parts (twice). We have

$$\begin{aligned}\int x^2\sin(nx)\,\mathrm{d}x &= -\frac{x^2\cos(nx)}{n} - \int -\frac{2x\cos(nx)}{n}\,\mathrm{d}x\\ &= -\frac{x^2\cos(nx)}{n} + \frac{2}{n}\int x\cos(nx)\,\mathrm{d}x,\end{aligned}$$

by letting $u = x^2$ and $\mathrm{d}v = \sin(nx)\,\mathrm{d}x$, so that $\mathrm{d}u = 2x\,\mathrm{d}x$ and $v = -\cos(nx)/n$. For the next step, we continue to

$$\int x^2\sin(nx)\,\mathrm{d}x = -\frac{x^2\cos(nx)}{n} + \frac{2}{n}\left(\frac{x\sin(nx)}{n} - \int\frac{\sin(nx)}{n}\,\mathrm{d}x\right),$$

where in the second integration by parts the choices made where $u = x$ and $\mathrm{d}v = \cos(nx)/n$, leading to $\mathrm{d}u = \mathrm{d}x$ and $v = \sin(nx)/n$. Evaluating the last integral, we finally obtain

$$\int x^2\sin(nx)\,\mathrm{d}x = -\frac{x^2\cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3}.$$

Back to computing $b_n$'s, using that $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$, we get

$$\begin{aligned}b_n &= \frac{2}{\pi}\left(-\frac{x^2\cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3}\right)\Bigg|_0^\pi\\ &= \frac{2}{\pi}\left(-\frac{\pi^2(-1)^n}{n} + 0 + 0 - 0 + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right)\\ &= \frac{2}{n^3\pi}((-1)^n(2-n^2\pi^2) - 2).\end{aligned}$$

Hence, we have a solution to the heat flow problem given by

$$u(x,t) = \sum_{n\geq 1} \frac{2}{n^3\pi}((-1)^n(2-n^2\pi^2)-2)\mathrm{e}^{-3n^2t}\sin(nx).$$

## 15.2 The heat flow problem with insulated endpoints

The discussion below will not be part of exams or homework. It is meant to help you see the big picture regarding heat flow problems, and it is the last "new" material to be included in this class diary.

There are several variants of the heat flow problem we have studied so far, which are dealt with by using separation of variables. For example, consider the situation in which the endpoints of the wire considered do not have their temperature fixed at zero, but instead are *insulated* – that is, no heat enters or leaves the wire through its endpoints. We are led to

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \beta\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0<x<L,\ t>0 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0, & t>0 \\ u(x,0) = f(x), & 0<x<L \end{cases}$$

That is, the insulation condition translated on a homogeneous boundary condition for $\partial u/\partial x$ instead of $u$.

Very briefly, consider again the separation of variables $u(x,t) = X(x)T(t)$, so that the partial differential equation becomes $X(x)T'(t) = \beta X''(x)T(t)$, leading to

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\beta T(t)} = -\lambda.$$

The resulting differential equations $X''(x)+\lambda X(x) = 0$ and $T'(t)+\lambda\beta T(t) = 0$ remain completely unchanged, but instead of the boundary conditions $X(0) = X(L) = 0$ from the previous heat flow problems we discussed, we have $X'(0) = X'(L) = 0$.

The eigenvalues of the boundary value problem

$$\begin{cases} X''(x)+\lambda X(x) = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

are again given by $\lambda_n = (n\pi/L)^2$, but the corresponding eigenfunctions are given by $X_n(x) = \cos(n\pi x/L)$ (instead of $\sin(n\pi x/L)$). We have a sequence of positive eigenvalues increasing to $+\infty$, and the new boundary condition is still homogeneous, so that the prototype solution for a heat flow problem with insulated endpoints is

$$u(x,t) = \frac{c_0}{2} + \sum_{n\geq 1} c_n\mathrm{e}^{-\beta(n\pi/L)^2t}\cos\left(\frac{n\pi x}{L}\right),$$

where the $c_n$'s are the coefficients of the Fourier *cosine* series of $f(x)$.

In other words, Fourier cosine series are also relevant to solving heat flow problems, it just turned out that in this class the only problem we had time to study in detail was the one with the endpoints of the wire had their temperatures fixed at zero. With similar ideas, one may study a heat flow problem with prescribed temperatures $U_1$ and $U_2$ at the endpoints, such as

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \beta \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = U_1 \quad \text{and} \quad u(L,t) = U_2, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

**Example 41**

*Solve the heat flow problem with insulated endpoints*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 3\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < \pi,\ t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(\pi,t) = 0, & t > 0 \\ u(x,0) = x^2, & 0 < x < \pi \end{cases}$$

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This time, we may take

$$u(x,t) = \frac{c_0}{2} + \sum_{n\geq 1} c_n e^{-3n^2 t}\cos(nx)$$

as our starting point, and find the $c_n$'s that work. As

$$\left(\frac{c_0}{2} + \sum_{n\geq 1} c_n e^{-3n^2 t}\cos(nx)\right)\Bigg|_{t=0} = \frac{c_0}{2} + \sum_{n\geq 1} c_n \cos(nx) = x^2,$$

we see that the Fourier *cosine* series of $f(x) = x^2$ on $[0,\pi]$ is needed to continue. Switching the roles of sin and cos in the calculation done in Example 40, we can obtain (do it!)

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n\geq 1} \frac{4(-1)^n}{n^2}\cos(nx),$$

so that

$$u(x,t) = \frac{\pi^2}{3} + \sum_{n\geq 1} \frac{4(-1)^n}{n^2} e^{-3n^2 t}\cos(nx).$$

# 16 December 5th

Free review before the final exam.

# References

[1] **Math 2177, Third Custom Edition for OSU**, Pearson. ISBN 10: 0-13-720383-7.

[2] Kreider, Kuller, Ostberg, Perkins; **An Introduction to Linear Analysis**, Addison-Wesley Publishing Company, Inc., Reading, Massachussets, U.S.A., 1966.