Parallel differential forms of codegree two, and three-forms in dimension six

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Abstract. For a differential form on a manifold, having constant components in suitable local coordinates trivially implies being parallel relative to a torsion-free connection, and the converse implication is known to be true for p-forms in dimension n when $p = 0, 1, 2, n-1, n$. We prove the converse for $(n-2)$-forms, and for 3-forms when $n = 6$, while pointing out that it fails to hold for Cartan 3-forms on all simple Lie groups of dimensions $n \geq 8$ as well as for $(n, p) = (7, 3)$ and $(n, p) = (8, 4)$, where the 3-forms and 4-forms arise in compact simply connected Riemannian manifolds with exceptional holonomy groups. We also provide geometric characterizations of 3-forms in dimension six and $(n-2)$-forms in dimension n having the constant-components property mentioned above, and describe examples illustrating the fact that various parts of these geometric characterizations are logically independent.

1. Introduction

Manifolds (by definition connected) and tensor fields are always assumed to be smooth. A differential p-form μ on an n-dimensional manifold M may be called

(i) *algebraically constant* if it has the same algebraic type at all points,
(ii) *locally constant* when it has constant components in suitable local coordinates around each point,
(iii) *parallel* if $\nabla\mu = 0$ for some torsion-free connection ∇.

In [**8**] our 'parallel' forms are referred to as *integrable*, while [**15**] uses for (ii) the term *forms with constant coefficients.* With the aid of partitions of unity one easily sees that ∇ in (iii) only needs to exist locally. Thus,

$$\text{(1.1)} \qquad \text{(iii) follows from (ii), which is its local version with a flat connection,}$$

if we agree to treat the items (i)–(iii) as conditions rather than definitions, Including (1.1), we then have the well-known implications

$$\text{(1.2)} \qquad \text{(ii)} \implies \text{(iii)} \implies \text{(i)}, \quad \text{and} \quad \text{(iii)} \implies (d\mu = 0),$$

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cf. [**15**, Prop. 2.1], [**8**, formula (1.1)], as well as the equivalences

$$\text{when } p \in \{0,1,2,n-1,n\}, \quad \text{(ii)} \iff \text{(iii)} \iff (\mu \text{ is closed}). \tag{1.3}$$

See [**15**, Examples 1.5-1.8], [**8**, Prop. D]. As pointed out by Muñoz Masqué et al. in [**15**], the questions of this kind raised, but not answered, by (1.3) start from $(n,p)=(5,3)$. Regarding such questions, let us note here that the converse of the last implication in (1.2) is *false except in the case* (1.3): [**8**, formula (1.7)] provides simple counterexamples for all (n,p) with $n \geq 5$ and $3 \leq p \leq n-2$.

Given an algebraically constant differential form μ on a manifold and a distribution $\mathcal{E}$ naturally associated with μ, as $\mathcal{E}$ is obviously ∇-parallel when $\nabla\mu = 0$,

$$\text{integrability of } \mathcal{E} \text{ follows if } \mu \text{ happens to be parallel.} \tag{1.4}$$

Our first main result, Theorem 9.1, states that the equivalence (ii) $\iff$ (iii) in (1.3) remains true also when $p = n-2$, as well as for $(n,p)=(6,3)$.

In general, however, (ii) implies (iii), but not conversely, with specific counterexamples (Theorem 14.1): some related to exceptional holonomy groups [**12**], with $(n,p)=(7,3)$ and $(n,p)=(8,4)$, others provided by the Cartan 3-forms on all simple Lie groups of dimensions $n \geq 8$. The latter can be further generalized (Remark 14.2) to all semisimple Lie groups without normal Lie subgroups of dimensions 3 or 6.

The next two results, Theorems 9.2 and 9.3, provide geometric characterizations of the case where μ, a 3-form in dimension 6 or an $(n-2)$-form in dimension n, is parallel (or, equivalently, locally constant, cf. Theorem 9.1). The characterizations involve closedness of μ and, for $(n-2)$-forms, of a certain 2-form arising from μ, as well as integrability of an almost-complex structure (in the case of 3-forms) and of specific distributions naturally associated with μ.

In Sect. 15–16 we describe examples showing that there are no redundant parts in the characterizations just mentioned. One source of these examples is Theorem 16.1, dealing with the natural duality between nondegenerate differential 2-forms σ in (necessarily even) dimension n and $(n-2)$-forms μ which are indivisible at each point. It states that, even though $d\mu = 0$ whenever $d\sigma = 0$, the converse implication, obvious when $n \leq 4$, fails in all even dimensions $n \geq 6$.

2. Preliminaries

Our convention about the exterior product of 1-forms ξ^i and vectors v_j is

$$[\xi^1 \wedge \ldots \wedge \xi^p](v_1,\ldots,v_p) \,=\, [v_1 \wedge \ldots \wedge v_p](\xi^1,\ldots,\xi^p) \,=\, \det[\xi^i(v_j)]. \tag{2.1}$$

Given a differential p-form ζ, with $\widehat{\ }$ meaning 'delete' one has the following well-known expression for the exterior derivative in terms of Lie brackets:

$$\begin{aligned}[d\zeta](v_0,\ldots,v_p) \;=\; &\textstyle\sum_i (-1)^i d_{v_i}[\zeta(v_0,\ldots,\widehat{v}_i,\ldots,v_p)] \\ +\; &\textstyle\sum_{i,j} (-1)^{i+j} \zeta([v_i,v_j],v_0,\ldots,\widehat{v}_i,\ldots,\widehat{v}_j,\ldots,v_p),\end{aligned} \tag{2.2}$$

$v_0,\ldots,v_p$ being any tangent vector fields, the summation ranges $0 \leq i \leq p$ and and $0 \leq i < j \leq p$. See, e.g., [**2**, formula (1.5a)].

We call a vector field w on a manifold *projectable along an integrable distribution* $\mathcal{D}$ if, locally, it is projectable onto local leaf spaces of $\mathcal{D}$. As is easily seen in coordinates such that some coordinate vector fields span $\mathcal{D}$,

(2.3) w is projectable along $\mathcal{D}$ if and only if, for every section v of $\mathcal{D}$ the Lie bracket $[w,v]$ is also a section of $\mathcal{D}$.

One says that a $(0,r)$ tensor field ξ on a manifold M *annihilates* a distribution $\mathcal{D}$ (or, is *projectable along* $\mathcal{D}$) if $\xi(v_1,\dots,v_r)=0$ whenever one of the vector fields $v_1,\dots,v_r$ is a section of $\mathcal{D}$ or, respectively, if $\mathcal{D}$ is integrable, and ξ, locally, equals the pullback to M of a $(0,r)$ tensor field on a local leaf space of $\mathcal{D}$.

LEMMA 2.1. *Let $\mathcal{D}$ be a ∇-parallel distribution on a manifold M with a torsion-free connection ∇. If a $(0,r)$ tensor field ξ on M annihilates $\mathcal{D}$, then so does $\nabla_v\xi$, for any vector field v, while $\nabla_v\xi=0$ when, in addition, ξ is projectable along $\mathcal{D}$ and v is a section of $\mathcal{D}$.*

We get both claims evaluating $[\nabla_v\xi](v_1,\dots,v_r)$ from the Leibniz rule; the second, as $v_1,\dots,v_r$ may be assumed projectable along $\mathcal{D}$, so that $\xi(v_1,\dots,v_r)$ is constant along $\mathcal{D}$, while, by (2.3), each $[v,v_i]$, and hence $\nabla_v v_i$, is a section of $\mathcal{D}$.

LEMMA 2.2. *With the index ranges $j=1,\dots,r$ and $k=r+1,\dots,n$, let functions x^k on an n-dimensional manifold M have $dx^{r+1}\wedge\ldots\wedge dx^n\neq 0$ everywhere and be constant along mutually commuting vector fields e_j that are linearly independent at each point. Then, locally, there exist coordinates $x^1,\dots,x^n$ including our x^k for which e_j, $j=1,\dots,r$, are the coordinate vector fields ∂_j.*

PROOF. Fix a codimension r submanifold Q transverse to the span of our e_j. Let $F(z,x^1,\dots,x^r)=x(1)$ for $z\in Q$ and real $x^1,\dots,x^r$ such that $x(1)$ exists for the integral curve $t\mapsto x(t)$ of the combination x^je_j with $x(0)=z$. By the inverse mapping theorem, F provides a diffeomorphic identification of a neighborhood of any given point z in M with the Cartesian product of a neighborhood of z in Q and a neighborhood of zero in the Euclidean r-space. This turns the variables x^j into the required additional coordinate functions. □

LEMMA 2.3. *For a closed differential p-form η on a Cartesian-product manifold M with $TM=\mathcal{H}^+\oplus\mathcal{H}^-$ for the factor distributions $\mathcal{H}^\pm$, let $\eta=\eta^++\eta^-$, where each of the p-forms $\eta^\pm$ annihilates $\mathcal{H}^\mp$. Then both $\eta^\pm$ are the pullbacks to M of some closed p-forms on the factor manifolds.*

PROOF. Applying (2.2) to suitable coordinate vector fields for a Cartesian-product coordinate system we see that $d\eta^\pm=0$ and the component functions of $\eta^\pm$ are constant along $\mathcal{H}^\mp$, as required. □

REMARK 2.4. Locally in $\mathrm{I\!R}^n$, every function ϕ is the divergence of some vector field w, for instance, $w=(\psi,0,\dots,0)$ with ψ such that $\partial_1\psi=\phi$.

REMARK 2.5. Given an n-dimensional manifold M and a local trivialization $\xi^1,\dots,\xi^n$ of T^*M dual to a local trivialization $e_1,\dots,e_n$ of TM with functions C_{ij}^k such that $[e_i,e_j]=C_{ij}^ke_k$, one has, by (2.2) and (2.1), $[d\xi^k](e_i,e_j)=-C_{ij}^k$, and hence $d\xi^k=-C_{ij}^k\xi^i\wedge\xi^j$.

REMARK 2.6. If $p\geq 2$, any $(p,0)$ differential form ω on a complex manifold, having a closed real part, must itself be closed (that is, holomorphic). In fact, $d\omega=\partial\omega+\bar\partial\omega$ is then imaginary, and hence opposite to its conjugate $d\bar\omega$, while $d\bar\omega$ has bihomogeneous components of bidegrees $(0,p+1)$ and $(1,p)$, different of the bidegrees $(p+1,0)$ and $(p,1)$ for $d\omega$ unless $p\in\{0,1\}$.

3. Some invariants of exterior forms

Throughout this section V is a real vector space of dimension n. We call a p-vector $\beta \in V^{\wedge p}$ or an exterior p-form $\mu \in [V^*]^{\wedge p}$ *decomposable* if it is the exterior product of p vectors or 1-forms. A *volume form* in V is a nonzero exterior n-form, which amounts to a nonzero scalar when $n = 0$.

Let $\mu \in [V^*]^{\wedge p}$ be an exterior p-form in V, where $1 \le p \le n$. Its *rank* is the minimum dimension of a vector space W such that μ equals the pullback of an exterior p-form in W under some linear operator $V \to W$. Since $V \to W$ may be assumed surjective, when $r = \operatorname{rank}\mu$ and $\mu \ne 0$,

(3.1) $\quad p \ne r-1$ and $p \le r \le n$ with $p = r$ if and only if μ is decomposable,

due to the well-known decomposability of r-forms and $(r-1)$-forms in dimension r. See, for instance, [**13**, pp. 287-288], [**15**, Examples 1.6, 1.8] or [**8**, Sect. 11].

We associate with μ two vector subspaces of V. One is the *kernel* Z of μ, in other words, the kernel of the operator $V \to [V^*]^{\wedge(p-1)}$ sending v to $\mu(v, \cdot, \dots, \cdot)$. The other space, which we call the *divisibility space* of μ and denote by D, is the polar space (annihilator) of the subspace D' of $[V^*]$ consisting of 1-forms $\xi \in V^*$ such that μ is $\wedge$-divisible by ξ (or, equivalently, $\xi \wedge \mu = 0$). Thus, D is the simultaneous kernel of all such 1-forms ξ. Then, for $r = \operatorname{rank}\mu$ and $k = \dim Z$,

(3.2) $\quad$ a) $k = n - r$, $\quad$ b) $Z \subseteq D$ unless $\mu = 0$.

In fact, (3.2-a) follows since μ clearly equals the pullback under the projection operator $V \to V/Z$ of the exterior p-form in V/Z that μ descends to, the minimum-dimension clause being obvious as a pullback form vanishes on the kernel of the operator used to pull it back. To obtain (3.2-b), note that, if $v \in Z$ and $\xi \in D'$, one has $\xi(v)\mu(v_1, \dots, v_p) = [\xi \wedge \mu](v, v_1, \dots, v_p) = 0$ for all $v_1, \dots, v_p \in V$.

When $\mu = 0$ and $V \ne \{0\}$, (3.2-b) fails to hold: $Z = V$ and $D = \{0\}$. For nonzero scalars (0-forms) μ we set $Z = \{0\}$ and $D = V$. Generally,

(3.3) $\quad$ we call an exterior form μ *indivisible* if $D = V$,

that is, if $\xi \wedge \mu \ne 0$ whenever $\xi \in V^* \smallsetminus \{0\}$. We will repeatedly assume that

(3.4) $\quad \xi^1, \dots, \xi^n$ is the basis of V^* dual to a basis $e_1, \dots, e_n$ of V.

Then, by (2.1), for $\xi = \mu(\,\cdot\,, e_{j_2}, \dots, e_{j_p})$ with $\mu = \xi^{i_1} \wedge \ldots \wedge \xi^{i_p} \ne 0$,

(3.5) $\quad \xi = \pm\xi^i$ if $\{i_1, \dots, i_p\} = \{i\} \cup \{j_2, \dots, j_p\}$, and $\xi = 0$ when $\{i_1, \dots, i_p\} \smallsetminus \{j_2, \dots, j_p\}$ is not a one-element set.

Lemma 3.1. *If $\xi^1, \dots, \xi^s \in V^*$ are linearly independent, any exterior form η with $\xi^1 \wedge \ldots \wedge \xi^s \wedge \eta = 0$ lies in the ideal generated by $\xi^1, \dots, \xi^s$.*

Proof. Let $\eta \in [V^*]^{\wedge p}$. Expanding η as a linear combination of the obvious basis of $[V^*]^{\wedge p}$ arising from a basis $\xi^1, \dots, \xi^n$ of V^* which includes our $\xi^1, \dots, \xi^s$, we see that the p-fold exterior products without any of the factors $\xi^1, \dots, \xi^s$ occurring in the expansion of η with nonzero coefficients would remain linearly independent even after being $\wedge$-multiplied by $\xi^1 \wedge \ldots \wedge \xi^s$. As $\xi^1 \wedge \ldots \wedge \xi^s \wedge \eta = 0$, there are no p-fold products with the above properties. $\square$

Remark 3.2. The *image* of $\mu \in [V^*]^{\wedge p}$, defined to be the span in V^* of all $\mu(\,\cdot\,, v_2, \dots, v_p)$ for $v_2, \dots, v_p \in V$. Obviously,

(3.6) the image of μ is the polar space of $\operatorname{Ker}\mu$, so that, by (3.2-a), the dimension of the image of μ equals $\operatorname{rank}\mu$

Assuming (3.4), we easily see that

(3.7) the image of μ is spanned by $\{\mu(\,\cdot\,, e_{j_2}, \dots, e_{j_p}) : 1 \le j_2 < \dots < j_p \le n\}$

and so, as a consequence of (3.5), the image of μ is then

(3.8) contained in the span of all ξ^i occurring in μ.

The word 'occurring' means here that the expansion of μ as a linear combination of the obvious basis of $[V^*]^{\wedge p}$ includes, with a nonzero coefficient, a p-fold exterior product involving the factor ξ^i.

Remark 3.3. Under the assumption (3.4), we will say that a p-element set $I \subseteq \{1, \dots, n\}$ *occurs in* an exterior p-form μ if the expansion of μ as a linear combination of the obvious basis of $[V^*]^{\wedge p}$ includes $\xi^{i_1} \wedge \dots \wedge \xi^{i_p}$ with a nonzero coefficient, where $I = \{i_1, \dots, i_p\}$. Let S be the union of the distinct p-element subsets $I_1, \dots, I_l$ of $\{1, \dots, n\}$ occuring in μ. If $p \ge 2$ and any $(p-1)$-element subset of $\{1, \dots, n\}$ is contained in at most one of $I_1, \dots, I_l$ (for instance, $I_1, \dots, I_l$ are pairwise disjoint), then the image of μ is the span of $\{\xi^i : i \in S\}$ (and hence, by (3.6), $\operatorname{rank}\mu = |S|$, so that $\operatorname{rank}\mu = p$ if μ is decomposable, with $l = 1$). In fact, one inclusion is provided by (3.8). For the other one, we fix ξ^i with $i \in S$ and apply (3.5) to $j_2, \dots, j_p$ such that $\{i\} \cup \{j_2, \dots, j_p\}$ is one of $I_1, \dots, I_l$.

Remark 3.4. The kernel of any nonzero decomposable p-form μ coincides with its divisibility space: writing $\mu = \xi^1 \wedge \dots \wedge \xi^p$, with (3.4), we see that both have the same polar space $\operatorname{Span}(\xi^1, \dots, \xi^p)$. (The former according to (3.6) and Remark 3.3, the latter since the equality $\xi^1 \wedge \dots \wedge \xi^p \wedge \xi = 0$ for a 1-form ξ amounts to linear dependence of the system $\xi^1, \dots, \xi^p, \xi$.)

Remark 3.5. For $p \ge 2$ and linearly independent 1-forms $\xi^1, \dots, \xi^{p+2} \in V^*$, the p-form $\mu = (\xi^1 \wedge \xi^2 + \xi^3 \wedge \xi^4) \wedge \xi^5 \wedge \dots \wedge \xi^{p+2}$ is not decomposable: Remark 3.3 gives $\operatorname{rank}\mu = p + 2$ for our μ, and rank equal to p for decomposable p-forms.

Lemma 3.6. *The divisibility space of an exterior p-form $\mu = \theta \wedge \zeta$ is $\{0\} \times D$ whenever θ and ζ are the pullbacks to the direct-product vector space $V = W \times D$ of a volume form in W and an indivisible exterior form in D.*

Proof. For $e_1, \dots, e_n$ with (3.4) having the first s vectors in $W \times \{0\}$ and the last $n - s$ in $\{0\} \times D$, such that $\theta = \xi^1 \wedge \dots \wedge \xi^s$, expanding $\xi \wedge \mu$ as a linear combination of the obvious basis of $[V^*]^{\wedge p}$, where $\xi = a_i \xi^i$, and "canceling" the factor $\xi^1 \wedge \dots \wedge \xi^s$ in each nonzero term of the expansion, we see that $\xi \wedge \mu = 0$ if and only if ζ is $\wedge$-divisible by $a_{s+1}\xi^{s+1} + \dots + a_n \xi^n$ when viewed as a form in D, which amounts to $a_{s+1} = \dots = a_n = 0$. □

4. Further invariants

As before, V is a real vector space of dimension n. The following lemma may be thought of as a converse of Lemma 3.6.

Lemma 4.1. *Let* $\mu \in [V^*]^{\wedge p}$ *be a nonzero exterior* p*-form of rank* r *in* V, *with* $k = n - r = \dim Z$ *and* $s = n - \dim D \le r$ *for the kernel* Z *and divisibility space* D *of* μ, *cf.* (3.2). *This has four consequences.*

(a) $s \ne p - 1$ *and* $s \le p$, *with equality if and only if* μ *is decomposable.*
(b) *For any basis* $\xi^1, \dots, \xi^n$ *of* V^* *such that* $\xi^1, \dots, \xi^s$ *is a basis of* D', *the polar space of* D, *one has* $\mu = \xi^1 \wedge \dots \wedge \xi^s \wedge \zeta$, *where the indivisible exterior* $(p-s)$*-form* ζ *is a linear combination of* $(p-s)$*-factor exterior products of* 1*-forms from the set* $\{\xi^{s+1}, \dots, \xi^n\}$.
(c) *If* $\mu = \xi^1 \wedge \dots \wedge \xi^s \wedge \zeta$ *for some basis* $\xi^1, \dots, \xi^s$ *of* D' *and some exterior* $(p-s)$*-form* ζ, *then the restriction of any such* ζ *to* D *is, uniquely, up to a nonzero scalar factor, determined by* μ.
(d) *The above restriction of* ζ *to* D *is indivisible in* D.

Proof. Given a basis $\xi^1, \dots, \xi^n$ as in (b), μ is a nonzero-coefficients linear combination of several exterior products $\xi^{i_1} \wedge \dots \wedge \xi^{i_q}$ with $i_1 < \dots < i_q$. The equalities $\xi^i \wedge \mu = 0$ for $i = 1, \dots, s$ amount to $\{1, \dots, s\} \subseteq \{i_1, \dots, i_q\}$ for each $\xi^{i_1} \wedge \dots \wedge \xi^{i_q}$ present in the combination, leading to the required decomposition $\mu = \xi^1 \wedge \dots \wedge \xi^s \wedge \zeta$, where ζ arises by "canceling" the factor $\xi^1 \wedge \dots \wedge \xi^s$ in each (nonzero) term of our expansion of μ, so that ζ is "built" from $\xi^{s+1}, \dots, \xi^n$. Any 1-form ξ, $\wedge$-dividing ζ, also divides μ, and so $\xi = a_1\xi^1 + \dots + a_s\xi^s$ for some $a_1, \dots, a_s$. Since $\xi^1, \dots, \xi^s$ are not present in ζ, writing $\xi \wedge \zeta = 0$ we see that $a_1 = \dots = a_s = 0$, and (b) follows. So does (a): if our ζ were *a* 1*-form*, obviously $\wedge$-dividing μ, it would have to be a linear combination of $\xi^1, \dots, \xi^s$ rather than being built from $\xi^{s+1}, \dots, \xi^n$.

Next, let $\mu = \xi^1 \wedge \dots \wedge \xi^s \wedge \zeta = \xi^1 \wedge \dots \wedge \xi^s \wedge \zeta'$, as in (c). Lemma 3.1 for $\eta = \zeta - \zeta'$, combined with uniqueness of $\xi^1 \wedge \dots \wedge \xi^s$ up to a factor, yields (c).

For (d), choose ζ as in (b). The restrictions of $\xi^{s+1}, \dots, \xi^n$ to D form a basis of D^*. Any linear combination of these restrictions, $\wedge$-dividing ζ, thus $\wedge$-divides μ, which makes it also a linear combination of $\xi^1, \dots, \xi^s$, and hence zero. □

We will refer to the restriction to D of the exterior $(p-s)$-form ζ in part (c) of Lemma 4.1 as an *indivisible factor* of the exterior p-form μ in V, and to $\theta = \xi^1 \wedge \dots \wedge \xi^s$ in (c) as a *volume factor* of μ. Since $(n-s) - (p-s) = n - p$,

(4.1) an indivisible factor of μ has the same codegree in D as μ does in V.

Also, $\xi^1, \dots, \xi^s$ descend to a basis of V/D, and so

(4.2) $\theta = \xi^1 \wedge \dots \wedge \xi^s$ descends to a volume form in V/D.

When $s = p$ (which is the decomposable case in Lemma 4.1(a)) we can obviously make ζ unique, by setting $\zeta = 1$.

Remark 4.2. If $n = 6$ and $\xi^1, \dots, \xi^6$ is a basis of V^*, consider the equality

$$\begin{aligned} \xi^1 \wedge \xi^2 \wedge \xi^3 + \xi^3 \wedge \xi^4 \wedge \xi^5 + \xi^5 \wedge \xi^6 \wedge \xi^1 \\ = \hat\xi^1 \wedge \hat\xi^2 \wedge \hat\xi^3 + \hat\xi^3 \wedge \hat\xi^4 \wedge \hat\xi^5 + \hat\xi^5 \wedge \hat\xi^6 \wedge \hat\xi^1 \end{aligned} \tag{4.3}$$

for some $\hat\xi^1, \dots, \hat\xi^6 \in V^*$ (which must then form a basis since, according to Remark 3.3, the 3-form on the left-hand side has rank six, while linear dependence of $\hat\xi^1, \dots, \hat\xi^6$ would make the rank of the right-hand side less than six, due to the original definition of rank at the beginning of this section). In such $\hat\xi^1, \dots, \hat\xi^6$,

(a) $\hat\xi^1, \hat\xi^3, \hat\xi^5$ can be any triple with $\mathrm{Span}(\hat\xi^1, \hat\xi^3, \hat\xi^5) = \mathrm{Span}(\xi^1, \xi^3, \xi^5)$, or

(b) $\hat\xi^2$ may be any nonzero 1-form in $\mathrm{Span}(\xi^2, \xi^4, \xi^6)$.

In fact, we get (a) by substituting for ξ^1, ξ^3, ξ^5 in (4.3) arbitrary linearly independent linear combinations of $\hat\xi^1, \hat\xi^3, \hat\xi^5$ and gathering terms which have the form $\hat\xi^j \wedge \ldots \wedge \hat\xi^k$ for (j,k) equal to $(1,3)$, $(3,5)$ and $(5,1)$. To obtain (b), note that the above process replaces ξ^2 with $\hat\xi^2 = c_5\xi^2 + c_1\xi^4 + c_3\xi^6$, where $(c_1, c_3, c_5) \in \mathrm{I\!R}^3$ is the vector product of the first two rows of the 3×3 matrix B satisfying the matrix equality $[\xi^1\ \ \xi^3\ \ \xi^5] = [\hat\xi^1\ \ \hat\xi^3\ \ \hat\xi^5]B$. Any prescribed nonzero vector product is realized in this way by some nonsingular 3×3 matrix.

Lemma 4.3. *Given an exterior 3-form μ in a six-dimensional real vector space V and a basis $\xi^1, \ldots, \xi^6$ of V^*, let H' be the set of all 1-forms $\xi \in V^*$ such that the 4-form $\xi \wedge \mu$ is decomposable.*

(i) *If $\mu = \xi^1\wedge\xi^2\wedge\xi^3 + \xi^3\wedge\xi^4\wedge\xi^5 + \xi^5\wedge\xi^6\wedge\xi^1$, H' equals the vector subspace of V^* spanned by ξ^1, ξ^3, ξ^5.*

(ii) *When $\mu = \xi^1\wedge\xi^2\wedge\xi^3 + \xi^4\wedge\xi^5\wedge\xi^6$, our H' is the set-theoretical union of two vector subspaces of V^*, the spans of ξ^1, ξ^2, ξ^3 and ξ^4, ξ^5, ξ^6.*

Proof. In (i), or (ii), $\xi\wedge\mu$ is decomposable, for a 1-form $\xi = a_i\xi^i$, if and only if $(a_2, a_4, a_6) = (0,0,0)$ or, respectively, one of (a_1, a_2, a_3), (a_4, a_5, a_6) equals $(0,0,0)$. Namely, the ‘if’ part is easily verified in both cases.

For the converse, in (i), let $(a_2, a_4, a_6) \neq (0,0,0)$. Remark 4.2(b) allows us to assume that $\xi = \xi^2 + a_1\xi^1 + a_3\xi^3 + a_5\xi^5$, and so

$$\xi\wedge\mu = \xi^2\wedge\xi^5\wedge(\xi^3\wedge\xi^4 + \xi^6\wedge\xi^1) + (a_1\xi^4 + a_3\xi^6 + a_5\xi^2)\wedge\xi^1\wedge\xi^3\wedge\xi^5.$$

In terms of the basis $e_1, \ldots, e_6$ of V dual to our basis $\xi^1, \ldots, \xi^6$ of V^*, if we now set $\eta_{ijk} = [\xi\wedge\mu](\,\cdot\,, e_i, e_j, e_k)$, (3.5) will give $\eta_{234} = \xi^5$, $\eta_{245} = \xi^3$ and $\eta_{256} = \xi^1$, as well as $\eta_{354} = \xi^2 + a_1\xi^1$ and $\eta_{253} = \xi^4 + a_5\xi^1$. Thus, by (3.6), $\mathrm{rank}\,[\xi\wedge\mu] \geq 5$ and $\xi\wedge\mu$ is not decomposable: if it were, it would have rank four (Remark 3.3).

In (ii), let $\hat\xi^1 = a_1\xi^1 + a_2\xi^2 + a_3\xi^3$ and $\hat\xi^4 = a_4\xi^4 + a_5\xi^5 + a_6\xi^6$ be both nonzero, and choose $\hat\xi^i$ for $i \in \{2,3,5,6\}$ with $\xi^1\wedge\xi^2\wedge\xi^3 = \hat\xi^1\wedge\hat\xi^2\wedge\hat\xi^3$ and $\xi^4\wedge\xi^5\wedge\xi^6 = \hat\xi^4\wedge\hat\xi^5\wedge\hat\xi^6$. Then $\xi\wedge\mu = (\hat\xi^1 + \hat\xi^4)\wedge\mu = \hat\xi^1\wedge\hat\xi^4\wedge(\hat\xi^5\wedge\hat\xi^6 - \hat\xi^2\wedge\hat\xi^3)$ is not decomposable as a consequence of Remark 3.5. ☐

5. The Hodge star duality

Again, V denotes a real vector space of dimension n.

We use the multiplicative notation $\mu\beta = \beta\mu$ for the natural bilinear pairing which associates with $\mu \in [V^*]^{\wedge p}$ and $\beta \in V^{\wedge \ell}$ the result of contractions of μ against β involving the maximum possible number of initial indices. Thus, $\mu\beta$ is a $(p-\ell)$-form if $p \geq \ell$, an $(\ell - p)$-vector when $\ell \geq p$, and hence a scalar in the case $\ell = p$, and, when $\beta = v_1\wedge\ldots\wedge v_\ell$ and $p \geq \ell$ (or, $\mu = \xi^1\wedge\ldots\wedge\xi^p$ and $\ell \geq p$), $\mu\beta = \mu(v_1, \ldots, v_\ell, \,\cdot\,, \ldots, \,\cdot\,)$ or, respectively, $\mu\beta = \beta\mu = \beta(\xi^1, \ldots, \xi^p, \,\cdot\,, \ldots, \,\cdot\,)$. Note that μv is the interior product $\imath_v\mu = \mu(v, \,\cdot\,, \ldots, \,\cdot\,)$ for $p \geq 1$ and $v \in V$.

For an ℓ-vector β, a p-form μ and a p'-form μ' one has the associative law

$$[\beta\mu]\mu' = \beta[\mu\wedge\mu'] \text{ if } p + p' \leq \ell. \tag{5.1}$$

which are obvious due to (2.1) under the assumption that

(5.2) the bases $v_1, \ldots, v_n$, of V and $\xi^1, \ldots, \xi^n$ of V^* are each other's duals, and $\beta = v_1 \wedge \ldots \wedge v_\ell$, while $\mu = \xi^1 \wedge \ldots \wedge \xi^p$ and $\mu' = \xi^{p+1} \wedge \ldots \wedge \xi^{p+p'}$. By skew-symmetry, (5.1) thus follows for β, μ, μ' that are all decomposable into factors from the bases (5.2), and the general case is immediate from trilinearity.

We now fix a volume n-form ω in V and the corresponding reciprocal n-vector α, with $\alpha\omega = 1$. By (2.1), in the case (5.2), if $\alpha = v_1 \wedge \ldots \wedge v_n$, then $\omega = \xi^1 \wedge \ldots \wedge \xi^n$. The *Hodge star isomorphisms* $\mu \mapsto *\mu$ and $\beta \mapsto *\beta$ of $[V^*]^{\wedge p}$ onto $V^{\wedge(n-p)}$, and vice versa, depending on ω, are then given by $*\mu = \alpha\mu$ and $*\beta = \omega\beta$. By (5.1),

$$*[\xi^1 \wedge \ldots \wedge \xi^p] = [v_1 \wedge \ldots \wedge v_n][\xi^1 \wedge \ldots \wedge \xi^p] = v_{p+1} \wedge \ldots \wedge v_n$$

with (5.2): both sides agree on any $(n-p)$-tuple $\xi^{i_{p+1}} \wedge \ldots \wedge \xi^{i_n}$ with $i_{p+1} < \ldots < i_n$ (as they equal 1 for $(i_{p+1}, \ldots, i_n) = (p+1, \ldots, n)$ and 0 otherwise). Evenly permuting the ξ^i and, simultaneously, the v_j, we now get, in the case (5.2),

$$(5.3) \qquad *[\xi^{i_1} \wedge \ldots \wedge \xi^{i_p}] = v_{i_{p+1}} \wedge \ldots \wedge v_{i_n}, \quad *[v_{i_1} \wedge \ldots \wedge v_{i_\ell}] = \xi^{i_{\ell+1}} \wedge \ldots \wedge \xi^{i_n}$$

whenever $i_1, \ldots, i_n$ is an even permutation of $1, \ldots, n$, the second equality immediate from the first when one switches the roles of V and V^*. Hence

$$(5.4) \qquad \begin{array}{l} * : V^{\wedge(n-p)} \to [V^*]^{\wedge p} \text{ equals } (-1)^{(n-p)p} \\ \text{times the inverse of } * : [V^*]^{\wedge p} \to V^{\wedge(n-p)}. \end{array}$$

With the volume n-form ω still fixed, let $\beta = *\mu$. Then

$$(5.5) \qquad \begin{array}{l} \text{the image of } \beta \text{ is the divisibility space of } \mu, \text{ and} \\ \text{the divisibility space of } \beta \text{ equals the kernel of } \mu, \end{array}$$

the two spaces associated with an ℓ-vector β being defined in the obvious way: $\{\beta\mu' : \mu' \in [V^*]^{\wedge(\ell-1)}\}$ and $\{v \in V : v \wedge \beta = 0\}$. In fact, obviously,

$$(5.6) \qquad \text{the image of } \beta \text{ is polar to its kernel } \{\xi \in V^* : \beta(\xi, \cdot, \ldots, \cdot) = 0\}.$$

Now (5.1) for α rather than β and $p' = 1$ yields the first line of (5.5) by showing that the two spaces in question have the same polar space, and the first line then clearly follows if one switches V with V^*.

If a 2-form σ in V is nondegenerate ($\operatorname{Ker}\sigma = \{0\}$), and so $n = 2m$ is even,

$$(5.7) \qquad \begin{array}{l} \text{we define } \textit{the dual} \text{ of } \sigma \text{ to be the } (n-2)\text{-form } \mu = \omega\beta \text{ for the vol-} \\ \text{ume form } \omega = (m!)^{-1}\sigma^{\wedge m}, \text{ where } \beta \text{ is the bivector reciprocal to } \sigma. \end{array}$$

Thus, under the assumption (3.4),

$$(5.8) \qquad \begin{array}{l} \text{the dual of } \sigma = \sigma_1 + \ldots + \sigma_m \text{ with } \sigma_i = \xi^{2i-1} \wedge \xi^{2i} \text{ equals} \\ \mu = \mu_1 + \ldots + \mu_m \text{ for } \mu_i = -\sigma_1 \wedge \ldots \wedge \sigma_{i-1} \wedge \sigma_{i+1} \wedge \ldots \wedge \sigma_m. \end{array}$$

In fact, the bivector reciprocal to σ is $\beta = -e_1 \wedge e_2 - \ldots - e_{n-1} \wedge e_n$, so that $\omega\beta = \mu$ for $\omega = (m!)^{-1}\sigma^{\wedge m} = \xi^1 \wedge \ldots \wedge \xi^n$, as $\omega[e_{2i-1} \wedge e_i] = -\mu_i$. By (5.8),

$$(5.9) \qquad \text{the dual of } \sigma \text{ equals } -1 \text{ if } n = 2 \text{ and } -\sigma \text{ when } n = 4.$$

REMARK 5.1. Let μ be a nonzero exterior $(n-2)$-form in an n-dimensional real vector space V. For the kernel Z and divisibility space D of μ, (3.1) and (3.2-a) give $k = \dim Z \in \{0, 2\}$, whereas $q = \dim D$ is even and $2 \le q \le n$ as a consequence of (5.5) and Lemma 4.1(a) for $p = n-2$ and $s = n-q$.

LEMMA 5.2. *Any $(n-2)$-form in V, with $n = 2m$, dual to a nondegenerate 2-form, is indivisible, as in (3.3). Conversely, any indivisible $(n-2)$-form μ is equal or opposite to the dual of a nondegenerate 2-form σ, and μ determines such*

σ uniquely up to a sign. For even m, the phrases 'or opposite to' and 'up to a sign' may be deleted. Furthermore, $\pm\sigma$ has an explicit expression in terms of μ.

Proof. With the index ranges $i = 1,\dots,m$ and $k = 1,\dots,n = 2m$, let μ be dual to σ. For $\xi = a_k\xi^k \in V^*$ and $\theta_k = \xi^1\wedge\ldots\wedge\xi^{k-1}\wedge\xi^{k+1}\wedge\ldots\wedge\xi^n$, where we use (5.8), $\xi\wedge\mu$ equals the combination of θ_{2i} and θ_{2i-1} with the coefficients a_{2i-1} and a_{2i}. Linear independence of all θ_k now gives $\xi = 0$ whenever $\xi\wedge\mu = 0$, proving our first claim. Also, μ uniquely determines the reciprocal β of σ, and hence σ itself, up to a nonzero scalar factor: if $\mu = \omega\beta = *\beta$ for *any* volume form ω, (5.4) with $p = 2$ gives $\beta = *\mu$.

Conversely, let $\mu \in [V^*]^{\wedge(n-2)}$ be indivisible. With ω and α as in the lines preceding (5.2), $\alpha\mu$ is – by (5.5) – a nondegenerate bivector, and so it has a reciprocal nondegenerate 2-form λ. Hence $(m!)^{-1}\lambda^{\wedge m}$ equals $\kappa\omega$ for some $\kappa\in\mathrm{I\!R}$. Replacing ω with $\widetilde{\omega} = c\omega$ leads to $\widetilde{\alpha}$ and $\widetilde{\lambda}$ equal, respectively, to $c^{-1}\alpha$ and $c\lambda$, so that $(m!)^{-1}\widetilde{\lambda}^{\wedge m} = \widetilde{\kappa}\widetilde{\omega}$, where $\widetilde{\kappa} = c^{m-1}\kappa$. Some choice of c, unique up to a sign, now gives $|\widetilde{\kappa}| = 1$ and, if m is even, a unique c yields $\widetilde{\kappa} = 1$. The 2-form $\sigma = \widetilde{\lambda} = c\lambda$, for this c, has the reciprocal bivector $\beta = c^{-1}\alpha\mu$ and $(m!)^{-1}\sigma^{\wedge m} = \pm\widetilde{\omega} = \pm c\omega$. Thus, by (5.4), $\pm\widetilde{\omega}\beta = \pm\omega\alpha\mu = \pm\mu$ with the required sign $\pm$. □

By Lemma 5.2, indivisible $(n-2)$-forms exist only in even dimensions n.

We use the term *duality* for the natural bijective correspondence, established in Lemma 5.2, for even dimensions n, between nondegenerate exterior 2-forms σ and indivisible $(n-2)$-forms μ, with both σ,μ only defined up to a sign.

6. Exterior 3-forms in dimension six

The algebraic classification of exterior 3-forms μ in a six-dimensional real vector space V, known since Reichel's 1907 thesis [**16**], is copied here, with minor changes, from Bryant's paper [**4**, p. 599]: the possible (nonzero) types appear as

$$\text{(6.1)}\quad \begin{array}{ll} \text{a)} & \mu = \xi^1\wedge\xi^2\wedge\xi^3 + \xi^3\wedge\xi^4\wedge\xi^5 + \xi^5\wedge\xi^6\wedge\xi^1 + \xi^2\wedge\xi^4\wedge\xi^6, \\ \text{b)} & \mu = \xi^1\wedge\xi^2\wedge\xi^3 + \xi^3\wedge\xi^4\wedge\xi^5 + \xi^5\wedge\xi^6\wedge\xi^1, \\ \text{c)} & \mu = \xi^1\wedge\xi^2\wedge\xi^3 + \xi^4\wedge\xi^5\wedge\xi^6, \\ \text{d)} & \mu = \xi^1\wedge\xi^2\wedge\xi^3 + \xi^4\wedge\xi^5\wedge\xi^1, \\ \text{e)} & \mu = \xi^1\wedge\xi^2\wedge\xi^3 \end{array}$$

in some basis $\xi^1,\dots,\xi^6$ of V^*, dual to a basis $e_1,\dots,e_6$ of V. Following Hitchin [**11**, pp. 551-552], we call (6.1-a) and (6.1-c) the *complex/real stable cases.*

The five types (6.1) are illustrated by the diagrams on p. 11.

Each of the first four types (6.1) has an associated pair of invariants:

$$\text{(6.2)}\quad \text{a) } \pm J \text{ and } \mu(J\,\cdot\,,\,\cdot\,,\,\cdot\,),\quad \text{b) } H \text{ and } \Theta,\quad \text{c) } H^{\pm} \text{ and } \eta^{\pm},\quad \text{d) } D \text{ and } \mathrm{I\!R}\zeta,$$

reflecting their stabilizer groups; see the exposition by Bryant [**4**, p. 602, Remark 31]. For the reader's convenience, our presentation of (6.2) is self-contained. The invariant character of the objects in question is, in each case, due to their being uniquely determined by μ.

In the complex stable case (6.1-a), as pointed out by Hitchin [**11**, p. 552],

$$\text{(6.3)}\qquad \mu \text{ equals the real part of } \omega = (\xi^4 + i\xi^1)\wedge(\xi^6 + i\xi^3)\wedge(\xi^2 + i\xi^5).$$

Our basis $\xi^1,\dots,\xi^6$ of V^* is dual to a basis $e_1,\dots,e_6$ of V. Setting

$$\text{(6.4)}\qquad Je_4 = e_1,\qquad Je_6 = e_3,\qquad Je_2 = e_5,$$

we define a complex-structure tensor $J : V \to V$, making the factor complex 1-forms in (6.3) complex-linear, so that $\wedge$ in (6.3) is also the complex exterior product,

(6.5) ω in (6.3) is complex-trilinear, and so $\mu(J\cdot\,, J\cdot\,,\,\cdot\,) = -\mu$, which in turn implies total skew-symmetry of $\mu(J\cdot\,,\,\cdot\,,\,\cdot\,)$.

As shown by Bryant [**4**, p. 596],

(6.6) $\pm J$ are the only complex-structure tensors with $\mu(J\cdot\,, J\cdot\,,\,\cdot\,) = -\mu$.

We now justify (6.6). Let $\mu_k = \mu(e_k,\,\cdot\,,\,\cdot\,)$ and $\xi^{jk} = \xi^j \wedge \xi^k$. From (6.1-a),

$$\mu_1 = \xi^{23} + \xi^{56}, \quad \mu_2 = \xi^{31} + \xi^{46}, \quad \mu_4 = \xi^{53} + \xi^{62}, \quad \mu_5 = \xi^{34} + \xi^{61}. \tag{6.7}$$

Assuming just that $\mu(J\cdot\,, J\cdot\,,\,\cdot\,) = -\mu$, we get $\mu(\,\cdot\,, J\cdot\,,\,\cdot\,) = \mu(J\cdot\,,\,\cdot\,,\,\cdot\,)$. Thus,

$$\mu(e_2, Je_1,\,\cdot\,) = \mu(Je_2, e_1,\,\cdot\,), \qquad \mu(e_4, Je_2,\,\cdot\,) = \mu(Je_4, e_2,\,\cdot\,), \tag{6.8}$$

and – for the same reason – as μ is skew-symmetric, $\mu(Jv, v,\,\cdot\,) = 0$ for any vector field v. Applied to the vector fields $e_1, \dots, e_6$, this implies that each Je_k is a section of $\operatorname{Ker}\mu_k$, and so, by (6.7), (3.6) and Remark 3.3, the spans of $\{e_1, e_4\}$ and $\{e_2, e_5\}$ are J-invariant. Writing $Je_1 = ae_1 + ce_4$, $Je_4 = \widetilde{c}e_1 + \widetilde{a}e_4$, and $Je_2 = a_{22}e_2 + a_{25}e_5$, then using (6.7) and (6.8), we obtain

$$-a\xi^3 + c\xi^6 = -a_{22}\xi^3 - a_{25}\xi^6, \qquad -a_{22}\xi^6 + a_{25}\xi^3 = \widetilde{c}\xi^3 - \widetilde{a}\xi^6.$$

Thus, $(\widetilde{a}, \widetilde{c}) = (a, -c)$ and the matrix of J in $\operatorname{Span}(e_1, e_4)$ has the rows $(a, -c)$, (c, a), making it conjugate to the multiplication by $a + ci$ in $\mathbb{C}$, so that $(a, c) = (0, \mp 1)$ and $Je_4 = \pm e_1$. As (6.1-a) is invariant under simultaneous cyclic permutations of $(1, 3, 5)$ and $(2, 4, 6)$, $Je_6 = \pm e_3$ and $Je_2 = \pm e_5$. The three $\pm$ signs must all be the same, since the last displayed formula, with $(a, c, \widetilde{a}, \widetilde{c}) = (0, \mp 1, 0, \pm 1)$ and (therefore) $a_{22} = 0$, gives $a_{25} = -c = \pm 1$. The third sign falls in line due to the aforementioned cyclic-permutation invariance, implying (6.6).

Next, (6.1-b) implies that, by Lemma 4.3(i), the subspace $H' = \operatorname{Span}(\xi^1, \xi^3, \xi^5)$ of V^* equals $\{\xi \in V^* : \xi \wedge \mu \text{ is decomposable}\}$. With $e_1, \dots, e_6$ dual to $\xi^1, \dots, \xi^6$ as before, we define $H = \operatorname{Span}(e_2, e_4, e_6) \subseteq V$ to be the polar space of H'. Setting $\varTheta v = \mu(v,\,\cdot\,,\,\cdot\,)$ we now obtain a linear isomorphism $\varTheta : H \to [H']^{\wedge 2}$. In fact, e_2, e_4, e_6 form a basis of H, while, by (6.1-b),

$$\varTheta e_2 = \xi^3 \wedge \xi^1, \qquad \varTheta e_4 = \xi^5 \wedge \xi^3 \qquad \varTheta e_6 = \xi^1 \wedge \xi^5. \tag{6.9}$$

Suppose next that we have (6.1-c). By Lemma 4.3(ii), the set of all 1-forms ξ such that $\xi \wedge \mu$ is decomposable is the union of the subspaces $\operatorname{Span}(\xi^1, \xi^2, \xi^3)$ and $\operatorname{Span}(\xi^4, \xi^5, \xi^6)$ of V^*. The resulting unordered pair $\{H^+, H^-\}$ of their polar spaces in V is therefore uniquely determined by μ and, consequently, so is the unordered pair $\{\eta^+, \eta^-\}$ of the 3-forms $\eta^+ = \xi^1 \wedge \xi^2 \wedge \xi^3$ and $\eta^- = \xi^4 \wedge \xi^5 \wedge \xi^6$. Note that $H^+ = \operatorname{Span}(e_4, e_5, e_6)$ and $H^- = \operatorname{Span}(e_1, e_2, e_3)$.

In the case (6.1-d), μ is not decomposable (Remark 3.5). The vector subspace D' of V^*, polar to the divisibility space D of μ, is thus the line spanned by ξ^1, as Lemma 4.1(a), with $p = 3$ and $s \geq 1$, gives $s = 1$. By (6.1-d), Lemma 4.1(c) now applies to $s = 1$ and $\zeta = \xi^2 \wedge \xi^3 + \xi^4 \wedge \xi^5$, turning $\zeta \in [D^*]^{\wedge 2}$ into an indivisible factor of μ, defined (up to multiplications by scalars) as in the lines preceding (4.2).

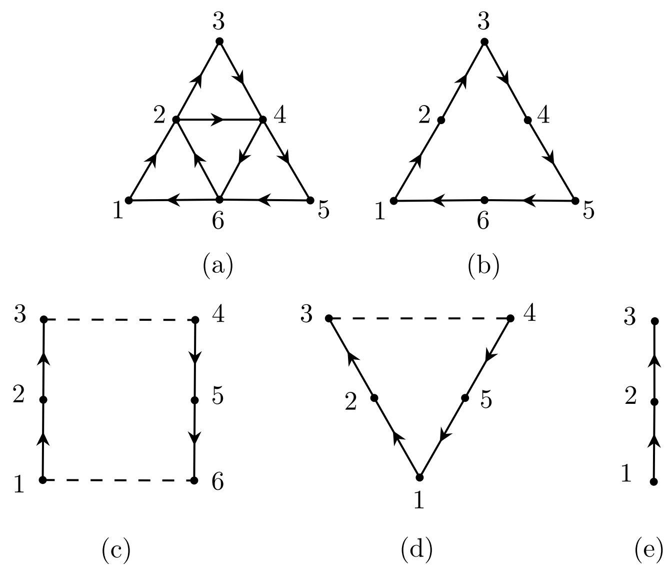

FIGURE 1. The five types (6.1) of 3-forms in dimension six. Each maximal solid line segment corresponds to one summand in (6.1), and so does the small inscribed $\bigtriangledown$ triangle in (a). They are all oriented as indicated by the arrows.

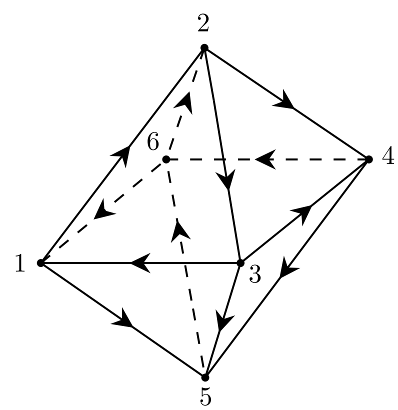

FIGURE 2. The octahedron version of (6.1-a). Four mutually non-adjacent faces correspond to $\mu = \mathrm{Re}\,\omega$ and the remaining four to $\mathrm{Im}\,\omega$, where seven faces (all but one of the latter, namely, 531) represent the same orientation of the boundary surface. The four μ-faces are also characterized by being coherently oriented by the arrow-marked orientations of their sides.

7. The simplest invariants of differential forms

Given a manifold M, we say that vector subbundles $\mathcal{D}$ of TM and $\mathcal{E}$ of T^*M are *polar to each other* when each is the other's "orthogonal complement" relative to the obvious pairing between tangent vectors and 1-forms.

For an algebraically constant differential p-form μ on an n-dimensional manifold M it obviously follows that, with $\mathcal{D}_x$ denoting the divisibility space of μ_x,

$$\text{(7.1)} \qquad \begin{array}{ll} \text{a)} & \text{the function } x \mapsto \operatorname{rank} \mu_x \text{ is constant on } M, \\ \text{b)} & \mathcal{D}_x \text{ has the same dimension } q \text{ at all } x \in M. \end{array}$$

The case (7.1-a), or (7.1-b), gives rise to the natural distribution $\mathcal{Z}$, or $\mathcal{D}$, on M, obtained by declaring $\mathcal{Z}_x$, or $\mathcal{D}_x$, at any $x \in M$, to be the subspace Z or, respectively, D of $V = T_xM$ associated with $\mu = \mu_x$ as in Sect. 3. We call $\mathcal{Z}$ and $\mathcal{D}$ the *kernel* and the *divisibility distribution* of μ. Thus, $\mathcal{D}$ is polar to the subbundle $\mathcal{D}'$ of T^*M having as sections the 1-forms ξ with $\xi \wedge \mu = 0$. Smoothness of $\mathcal{Z}$ and $\mathcal{D}$, under the respective assumptions (7.1-a) or (7.1-b), follows since $\mathcal{Z}$, or $\mathcal{D}'$, is the the kernel of a constant-rank vector-bundle morphism: the former, from TM to $[T^*M]^{\wedge(p-1)}$, sending v to $\mu(v, \cdot, \ldots, \cdot)$, the latter defined analogously, just with TM and T^*M switched; see (5.5) and (5.6). By (3.2-b),

$$\text{(7.2)} \qquad \text{if (7.1-a) and (7.1-b) both hold, } \mathcal{Z} \subseteq \mathcal{D} \text{ unless } \mu = 0.$$

When $d\mu = 0$, the distribution $\mathcal{Z}$ is easily seen to be integrable (Lemma 7.4 below). However, $\mathcal{D}$ need not be: see [**8**, Sect. 12].

By a *volume form* on a manifold we mean a nowhere-zero top degree differential form, which amounts to a function without zeros in dimension 0.

For a nonzero differential p-form μ satisfying (7.1-b) on an n-dimensional manifold M and its divisibility distribution $\mathcal{D}$, let $s = n - q$ in Lemma 4.1(a), so that $s \le p$ and, by Remark 5.1,

$$\text{(7.3)} \qquad \mathcal{D} \text{ has some even fibre dimension } q \in \{2, \ldots, n\} \text{ if } p = n - 2.$$

Whether or not $p = n - 2$, assuming integrability of $\mathcal{D}$, we now replace M with a sufficiently small neighborhood of any given point so as to make $\mathcal{D}$ the vertical distribution of a fibration $\pi : M \to \Sigma$, which gives rise to

$$\text{(7.4)} \qquad \begin{array}{l} \text{the } \pi\text{-pullback } \xi^1 \wedge \ldots \wedge \xi^s \text{ of a volume } s\text{-form on } \Sigma, \text{ for the} \\ \text{pull-backs } \xi^1, \ldots, \xi^s \text{ under } \pi \text{ of some 1-forms trivializing } T^*\Sigma, \\ \text{and then } \mu = \xi^1 \wedge \ldots \wedge \xi^s \wedge \zeta \text{ for some } (p-s)\text{-form } \zeta \text{ on } M, \end{array}$$

the last line due to Lemma 4.1(b) applied to μ and $\xi^1, \ldots, \xi^s$ at any point $x \in M$, with $D = \mathcal{D}_x$ and $V = T_xM$. By Lemma 4.1(c), the restriction of ζ to each leaf L of $\mathcal{D}$ is uniquely determined by our μ and $\xi^1, \ldots, \xi^s$. Replacing $\xi^1, \ldots, \xi^s$ by another such s-tuple of π-pullbacks causes $\xi^1 \wedge \ldots \wedge \xi^s$ to be replaced with its product by a function *constant along the leaves of* $\mathcal{D}$. Thus, the restriction of ζ to each leaf L is *unique up to multiplications by nonzero constants*, and

(7.5) we call this $(p-s)$-form ζ the *indivisible factor* of μ on the leaf L.

LEMMA 7.1. *For a nonzero closed differential p-form μ on an n-dimensional manifold M, satisfying the condition* (7.1-b) *and having the divisibility distribution $\mathcal{D}$ of codimension $s = n - q$, integrability of $\mathcal{D}$ implies closedness of the indivisible factor ζ of μ on every leaf of $\mathcal{D}$.*

PROOF. Since $\dim \Sigma = s$, a volume s-form on Σ chosen as in (7.4) is closed, leading to closedness of $\xi^1 \wedge \ldots \wedge \xi^s$ in (7.4). With $\mu = \xi^1 \wedge \ldots \wedge \xi^s \wedge \zeta$ as in the lines following (7.4), $\xi^1 \wedge \ldots \wedge \xi^s \wedge d\zeta = d\mu = 0$. Thus, by Lemma 3.1, $d\zeta$ lies in the ideal generated by $\xi^1, \ldots, \xi^s$, that is, $d\zeta = 0$ on each leaf of $\mathcal{D}$. □

Whenever a nonzero differential $(n-2)$-form μ with (7.1-b) on a manifold M dimension n is *indivisible* in the sense – cf. (3.3) – of having the divisibility distribution $\mathcal{D}$ equal to TM, Lemma 5.2 gives rise to a further invariant:

(7.6) *the 2-form dual to μ, locally unique up to a sign.*

More generally, for a differential $(n-2)$-form μ on an n-dimensional manifold M satisfying (7.1-b) and having an integrable divisibility distribution $\mathcal{D}$, we can, by (4.1), apply the last paragraph to any leaf L of $\mathcal{D}$, rather than M, and – instead of μ itself – to an indivisible factor ζ of μ on L mentioned in (7.5), obtaining

(7.7) a nondegenerate 2-form σ on L, unique up to multiplications by nonzero constants, which is dual to an indivisible factor of μ.

LEMMA 7.2. *If a nonzero differential $(n-2)$-form μ on an n-dimensional manifold M satisfies* (7.1-b) *and its divisibility distribution $\mathcal{D}$, having the fibre dimension q, is integrable, the bivectors on the leaves of $\mathcal{D}$ reciprocal to the nondegenerate 2-forms σ mentioned in* (7.7) *may be viewed, via an obvious push-forward, as forming a bivector β defined locally in M, and determined by μ uniquely up to multiplications by functions constant along $\mathcal{D}$. Then*

(a) *locally in M there exist 1-forms $\xi^{s+1},\dots,\xi^n$ with $\sigma=\xi^{s+1}\wedge\xi^{s+2}+\dots+\xi^{n-1}\wedge\xi^n$ along each leaf, $q=n-s$ being even due to* (7.3)*,*

(b) *for any 1-forms $\xi^1,\dots,\xi^s$ chosen as in* (7.4) *and any $\xi^{s+1},\dots,\xi^n$ as above, $\xi^1,\dots,\xi^n$ is a local trivialization of T^*M such that $\mu=\omega\beta$ for the volume form $\omega=\xi^1\wedge\dots\wedge\xi^n$.*

PROOF. Being nondegenerate, σ has the standard algebraic type [**5**, p. 13], and so $\xi^{s+1},\dots,\xi^n\in T_x^*M$ required in (a) exist at each point x, and may be augmented with s additional 1-forms in T_xM to constitute a basis of T_x^*M. Now (a) follows: $\xi^{s+1},\dots,\xi^n$ are just final portions of local trivializations of T^*M dual to local trivializations of TM that are the smooth local sections of a G-principal bundle over M, for a suitable matrix group $G\subseteq\mathrm{GL}(n,\mathbb{R})$. See, e.g., [**8**, Sect. 6].

For (b), note that the definition (5.7) of duality gives $\zeta=\theta\beta$ on each leaf L of $\mathcal{D}$, with M,m,μ and ω replaced by $L,q/2,\zeta$ and some suitable θ. By (7.4), $\mu=\xi^1\wedge\dots\wedge\xi^s\wedge[\beta\theta]$ and hence, as q is even, (5.1) yields $\omega\beta=\beta\omega=\beta[\theta\wedge\xi^1\wedge\dots\wedge\xi^s]=[\beta\theta]\wedge\xi^1\wedge\dots\wedge\xi^s=\mu$. □

The following obvious consequence of Lemma 3.6 will be used in Sect. 16.

REMARK 7.3. Given manifolds Π and Σ, a volume form θ on Σ, and an indivisible closed differential r-form ζ on Π satisfying the condition (7.1-b), let the symbols θ and ζ also stand for the corresponding pullback forms on the product manifold $M=\Pi\times\Sigma$. Then $\mu=\theta\wedge\zeta$ is a closed differential p-form on M with the property (7.1-b), for $p=r+s$ and $s=\dim\Sigma$, while the divisibility distribution $\mathcal{D}$ of μ is the factor distribution on M tangent to the Π factor manifold, and the restriction of ζ to $\mathcal{D}$ is the indivisible factor of μ.

LEMMA 7.4. *For a closed differential p-form μ with* (7.1-a) *on an n-manifold M, the kernel $\mathcal{Z}$ is integrable, and μ is projectable along $\mathcal{Z}$, in the sense of Sect.* 2*, onto a closed p-form on a local leaf space Σ.*

In fact, $\mathcal{Z}$ is integrable by (2.2). In local coordinates such that some of the coordinate fields ∂_i span $\mathcal{Z}$, (2.2) applied to $(p+1)$-tuples of ∂_i implies constancy along $\mathcal{Z}$ of the components of μ and closedness of the projected p-form.

Remark 7.5. The Darboux theorem with parameters. Let ζ be a constant-rank section of $[\mathcal{D}^*]^{\wedge 2}$ for an integrable distribution $\mathcal{D}$ of fibre dimension q on a manifold M. If the restriction of ζ to each leaf of $\mathcal{D}$ is closed, then, locally in M, and there exist functions $x^1,\dots,x^q$, constituting local coordinates on each leaf of $\mathcal{D}$, and such that $\zeta = dx^1\wedge dx^2 + \ldots + dx^{r-1}\wedge dx^r$, where r is the (even) rank of ζ. Namely, when $r=q$ this is [**1**, Lemma 3.10]. The general case follows: the vector subbundle $\operatorname{Ker}\zeta$ of $\mathcal{D}$ is integrable (Lemma 7.4), and we may replace M with a local leaf space of $\operatorname{Ker}\zeta$.

Remark 7.6. It is well known – see, e.g., [**15**, Example 1.6] or [**8**, Sect. 11] – that locally, in any dimension n, any given volume form equals $dx^1\wedge\ldots\wedge dx^n$ for suitable coordinates $x^1,\dots,x^n$. This remains true in the holomorphic category, with the same argument just cited from [**15**] or [**8**].

Remark 7.7. Let a decomposable differential p-form μ on a manifold be algebraically constant (that is, either identically zero, or nonzero everywhere). Then closedness of μ is equivalent to its local constancy, as well as to its parallelism. This is obvious from Remark 3.4, Lemma 7.4, and Remark 7.6 for $n=p$.

Remark 7.8. The condition (7.1-b) for a differential $(n-2)$-form μ in dimension n is equivalent to algebraic constancy of μ. Namely, in the lines following (7.4), $\mu=\xi^1\wedge\ldots\wedge\xi^s\wedge\zeta$, where s is the codimension of the divisibility distribution $\mathcal{D}$ and, by (4.1), the indivisible factor ζ restricted to $\mathcal{D}$ has codegree two. Being uniquely associated, via (7.7), with its dual 2-form σ, our ζ is thus algebraically constant due to nondegeneracy of σ.

8. Differential 3-forms in dimension six

For a nonzero algebraically constant differential 3-form on a 6-dimensional manifold M, each of the five cases of (6.1) is realized, locally, by

(8.1) smooth 1-forms $\xi^1,\dots,\xi^6$ trivializing T^*M, dual to a local trivialization $e_1,\dots,e_6$ of TM.

In fact, such $e_1,\dots,e_6$ are well known [**8**, Sect. 6] to be precisely the smooth local sections of a G-principal bundle over M, for some matrix group $G\subseteq \mathrm{GL}(n,\mathbb{R})$. Consequently, the invariants (6.2) give rise, locally, to analogous smooth objects in M, namely, an almost-complex structure J, the differential 3-forms η^+,η^- and $\mu(J\cdot,\cdot,\cdot)$, the distributions $\mathcal{H}$ and $\mathcal{H}^\pm$,

(8.2) the vector-bundle isomorphism $\Theta:\mathcal{H}\to[\mathcal{H}']^{\wedge 2}$,

the divisibility distribution $\mathcal{D}$ of μ, and finally – if, in addition, $\mathcal{D}$ is assumed to be integrable – the indivisible-factor 2-form ζ of μ defined, as in (7.5), along each leaf of $\mathcal{D}$, and only unique on the leaf up to multiplications by nonzero constants. Note that, in Sect. 6,

(8.3) $\mathcal{H}$ is spanned by e_2,e_4,e_6, and $\mathcal{H}^+$, $\mathcal{H}^-$ by e_4,e_5,e_6 and e_1,e_2,e_3.

9. Local constancy and parallelism of differential forms

The next result is immediate from Theorems 9.2 and 9.3, proved in Sect. 10–11.

Theorem 9.1. *The local constancy of an algebraically constant differential $(n-2)$-form on an n-dimensional manifold is equivalent to its being parallel.*
This is also the case for 3-forms in dimension six.

The analog of Theorem 9.1 is known [**8**, Prop. D] to hold for differential p-forms in dimension n, where $p \in \{0,1,2,n-1,n\}$. Thus, in dimensions $n \le 6$, a differential form of any degree is locally constant if and only if it is parallel. However, parallel forms that are not locally constant exist in infinitely many dimensions, starting from 7 and 8. See Theorem 14.1.

The objects $J, \mathcal{H}, \mathcal{H}^{\pm}, \mathcal{D}, \zeta$ in next theorem were described in Sect. 8.

Theorem 9.2. *The following three properties of a nonzero algebraically constant differential* 3*-form* μ *on a* 6*-dimensional manifold are mutually equivalent.*

(i) *Local constancy.*
(ii) *Being parallel.*
(iii) *Closedness of* μ*, coupled with*
  (a) *integrability of the almost-complex structure* J*, in case* (6.1-a)*,*
  (b) *integrability of the distribution* $\mathcal{H}$*, when* (6.1-b) *holds,*
  (c) *integrability of both* $\mathcal{H}^{\pm}$ *under the assumption* (6.1-c)*,*
  (d) *integrability of the divisibility distribution* $\mathcal{D}$*, for* (6.1-d)*,*
  (e) *no further condition in case* (6.1-e)*,*

The notions of indivisible factor and duality used below were defined in Sect. 7.

Theorem 9.3. *Given a nonzero algebraically constant differential* p*-form* μ *on an* n*-dimensional manifold* M*, with the divisibility distribution* $\mathcal{D}$ *and the indivisible factor* ζ*, the following two assumptions can be made about* μ*.*

(a) μ *is locally constant.*
(b) μ *is parallel.*

The condition (b) *always follows from* (a)*, while* (b) *implies that*

(i) μ *is closed and the distribution* $\mathcal{D}$ *is integrable.*

If $p = n-2$*,* (b) *has a further consequence, namely,*

(ii) *along each leaf of* $\mathcal{D}$*, the* 2*-form* σ *dual to* ζ *is closed.*

Conversely, for $p = n-2$*,* (i) *and* (ii) *together imply* (a)*, and hence* (b)*.*

10. Proof of Theorem 9.2

That (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is obvious from (1.2) and (1.4), since J in (iii-a), due to its naturality, is ∇-parallel when a torsion-free connection ∇ has $\nabla\mu = 0$. This last claim easily follows from the Newlander-Nirenberg theorem, as pointed out by various authors [**6**, Sect. 2.3], [**3**, Definition 2.2].

We now proceed to show that (iii) implies (i) by establishing, for suitable local coordinates $x^1, \dots, x^n$ and $\xi^1, \dots, \xi^6$ mentioned in (8.1),

(10.1) each of the five equalities (6.1) with every ξ^i replaced by dx^i.

First, in the case (6.1-a), ω given by (6.3) is, by (6.5), a complex volume $(3,0)$ form on the complex manifold M, so that ω must be holomorphic, due to closedness of its real part μ and Remark 2.6. The final clause of Remark 7.6 gives, locally, $\omega = dz^1 \wedge dz^2 \wedge dz^3$ in some holomorphic coordinates z^1, z^2, z^3. The real coordinates $x^1, \dots, x^6$ with $(z^1, z^2, z^3) = (x^4 + ix^1, x^6 + ix^3, x^2 + ix^5)$ now turn (6.3) into (10.1).

Next, assume (iii) and (6.1-b). The equality in (6.1-b) still holds, according to Remark 4.2(a), with suitable $\hat\xi^2, \hat\xi^4, \hat\xi^6$ instead of ξ^2, ξ^4, ξ^6, if one replaces ξ^1, ξ^3, ξ^5 with *any* local trivialization $\hat\xi^1, \hat\xi^3, \hat\xi^5$ of $\mathcal{H}'$, the vector subbundle of T^*M polar to the distribution $\mathcal{H}$ (and $\hat\xi^1, \dots, \hat\xi^6$ will then still, locally, trivialize T^*M). Due to integrability of $\mathcal{H}$, we are therefore free to choose the triple (ξ^1, ξ^3, ξ^5) in (6.1-b) equal to (dx^1, dx^3, dx^5), with some functions x^1, x^3, x^5 constant along the leaves of $\mathcal{H}$. For $e_1, \ldots, e_6$ dual to $\xi^1, \ldots, \xi^6$ as in (8.1), e_2, e_4, e_6 form a local trivialization of $\mathcal{H}$. Let the index ranges now be $i, j = 2, 4, 6$ and $k, l = 1, 3, 5$. In (6.9), $\Theta e_i = \mu(e_i, \cdot, \cdot)$ must thus be equal to the corresponding $\xi^k \wedge \xi^l = dx^k \wedge dx^l$, and consequently annihilate $\mathcal{H}$. Closedness of all $\xi^k = dx^k$ implies (see Remark 2.5) that all Lie brackets of $e_1, \ldots, e_6$ are tangent to $\mathcal{H}$. From the last two sentences and (2.2) with $d\mu = 0$ we now get

$$0 = -[d\mu](e_i, e_j, \cdot, \cdot) = \mu([e_i, e_j], \cdot, \cdot) = \Theta[e_i, e_j], \qquad i, j = 2, 4, 6.$$

Thus, due to the injectivity of Θ in (8.2), e_2, e_4, e_6 commute with one another, and Lemma 2.2 allows us to augment x^1, x^3, x^5 with three more functions so as to obtain, locally, a coordinate system $x^1, y^2, x^3, y^4, x^5, y^6$ for which e_i, $i = 2, 4, 6$, are the coordinate vector fields ∂_i. As $[dy^j](e_i) = [dy^j](\partial_i) = \delta_i^j = \xi^j(e_i)$, each $\xi^j - dy^j$, $j = 2, 4, 6$, annihilates $\mathcal{H}$, while $(\xi^1, \xi^3, \xi^5) = (dx^1, dx^3, dx^5)$. Therefore, ξ^2 (or ξ^4, or ξ^6) equals $dy^2 + \phi_5 dx^5$ (or $dy^4 + \phi_1 dx^1$, or $dy^6 + \phi_3 dx^3$) plus a functional combination of dx^1, dx^3 (or dx^3, dx^5 or, respectively, dx^1, dx^5), with some functions ϕ_1, ϕ_3, ϕ_5. Substituting the expressions just obtained for ξ^2, ξ^4, ξ^6, we rewrite (6.1-b) with $(\xi^1, \xi^3, \xi^5) = (dx^1, dx^3, dx^5)$ as

$$\begin{aligned} \mu \,=\, & \xi^1\wedge \xi^2\wedge \xi^3 + \xi^3\wedge \xi^4\wedge \xi^5 + \xi^5\wedge \xi^6\wedge \xi^1 \,=\, dx^1\wedge dy^2\wedge dx^3 \\ & + dx^3\wedge dy^4\wedge dx^5 + dx^5\wedge dy^6\wedge dx^1 - \phi\, dx^1\wedge dx^3\wedge dx^5, \end{aligned} \tag{10.2}$$

where $\phi = \phi_1 + \phi_3 + \phi_5$. Since μ is closed, (10.2) gives $d\phi \wedge dx^1\wedge dx^3\wedge dx^5 = 0$ and, by Lemma 3.1, ϕ is a function of the variables x^1, x^3, x^5, thus equal – see Remark 2.4 – to the divergence of some vector field $w = (w^1, w^3, w^5)$, with each w^k depending only on x^1, x^3, x^5. If we now set $(x^2, x^4, x^6) = (y^2 + w^5, y^4 + w^1, y^6 + w^3)$, (10.2) becomes (10.1) for the case (6.1-b), $x^1, \ldots, x^6$ being local coordinates as linear independence of $dx^1, \ldots, dx^6$ is immediate from the lines following (4.3).

Suppose now that (iii) and (6.1-c) hold. The subbundles $\mathcal{H}^\pm$ of TM, being integrable, are, locally, the factor distributions of a Cartesian-product decomposition of M, and so, by Lemma 2.3, $\eta^\pm$ are the pullbacks to M of some volume forms on the factor manifolds. Remark 7.6 now gives, locally, $\eta^+ = dx^1 \wedge dx^2 \wedge dx^3$ and $\eta^- = dx^4\wedge dx^5\wedge dx^6$ for some local coordinates x^1, x^2, x^3 and x^4, x^5, x^6 in the factors, proving (10.1) for (6.1-c).

For (iii-d), Lemma 7.1 and Remark 7.5 yield, locally, $\zeta = dx^2\wedge dx^3 + dx^4\wedge dx^5$ for suitable functions $x^2, \ldots, x^6$ constituting local coordinates on each leaf of $\mathcal{D}$, cf. (6.2-d), while ξ^1 then becomes the volume 1-form on Σ, appearing in (7.4) with $s = 1$. One-dimensionality of Σ now gives, locally, $\xi^1 = dx^1$ for some function x^1 with $dx^1 \neq 0$, constant along the leaves of $\mathcal{D}$, so that $\mu = \xi^1\wedge \zeta = dx^1\wedge (dx^2\wedge dx^3 + dx^4\wedge dx^5)$ in the resulting local coordinates $x^1, \ldots, x^6$, as required.

Finally, assuming (iii) and (6.1-e), we get (10.1) directly from Remark 7.7.

This completes the proof of Theorem 9.2.

11. Proof of Theorem 9.3

By (1.2) and (1.4), (a) $\Longrightarrow$ (b) $\Longrightarrow$ (i). Deriving (ii) from (b) requires a more subtle argument, the indivisible factor ζ of μ being defined, along each leaf of $\mathcal{D}$, only uniquely *up to multiplications by nonzero constants.* To this end, we assume (b). For torsion-free ∇ with $\nabla\mu = 0$, the 1-forms $\xi^1, \ldots, \xi^s$ in (7.4) annihilate the ∇-parallel distribution $\mathcal{D}$. By Lemma 2.1, ξ^i are all ∇-parallel along $\mathcal{D}$, and hence so is $\theta = \xi^1 \wedge \ldots \wedge \xi^s$. Thus, along each leaf L of $\mathcal{D}$, the restriction of ζ to L being, due to Lemma 4.1(c) uniquely determined by μ (and our fixed θ), is parallel relative to the torsion-free connection induced by ∇ on the totally geodesic submanifold L. The same then follows for the 2-form dual to the restriction of ζ, as the latter determines the former up to a sign (Lemma 5.2). Now (ii) follows.

To prove the final clause of the theorem, suppose now that a nonzero algebraically constant differential $(n-2)$-form μ on a manifold M of dimension n satisfies (i) and (ii). Being nondegenerate, the 2-form σ dual to ζ is, locally, a symplectic form on each leaf of $\mathcal{D}$. The Darboux theorem with parameters [**1**, Lemma 3.10] allows us, locally, to write $\sigma = dx^{s+1} \wedge dx^{s+2} + \ldots + dx^{n-1} \wedge dx^n$ for some functions $x^{s+1}, \ldots, x^n$ with $dx^{s+1} \wedge \ldots \wedge dx^n \neq 0$, where s is the codimension of $\mathcal{D}$. Any (local) functions $x^1, \ldots, x^s$ such that $dx^1 \wedge \ldots \wedge dx^s \neq 0$ and the leaves of $\mathcal{D}$ are the level sets of $(x^1, \ldots, x^s)$ give rise to local coordinates $x^1, \ldots, x^n$. The last $n-s$ of the corresponding coordinate vector fields ∂_i in M serve in the same capacity on leaves of $\mathcal{D}$, as $x^1, \ldots, x^s$ are constant along them, and so $\beta = -\partial_{s+1} \wedge \partial_{s+2} - \ldots - \partial_{n-1} \wedge \partial_n$ for the bivector β in Lemma 7.2. On the other hand, $dx^1, \ldots, dx^s$ may serve as $\xi^1, \ldots, \xi^s$ in (7.4), and Lemma 7.2(b) applied to $(\xi^1, \ldots, \xi^n) = (dx^1, \ldots, dx^n)$ gives $\mu = \omega\beta$ for the volume n-form $\omega = dx^1 \wedge \ldots \wedge dx^n$. Thus, the components of μ in the coordinates $x^1, \ldots, x^n$ are constant, as required.

12. Isotropy Lie algebras and connections

In a real vector space V of dimension n, any linear endomorphism $A \in \mathfrak{gl}(V)$ acts on $[V^*]^{\wedge p}$, for $1 \le p \le n$, as the derivation $\mu \mapsto A\mu$, with $[A\mu](v_1, \ldots, v_p)$ equal to the sum over $i = 1, \ldots, p$ of the terms $\mu(\widetilde{v}_1, \ldots, \widetilde{v}_p)$, where $\widetilde{v}_i = Av_i$ and $\widetilde{v}_j = v_j$ if $j \neq i$. Clearly, for the obvious action of $\mathrm{GL}(V)$ on $[V^*]^{\wedge p}$,

(12.1) $\mathfrak{h} = \{A \in \mathfrak{gl}(V) : A\mu = 0\}$ is the isotropy Lie algebra of μ.

Of particular interest to us is the case where

(12.2) In (12.1), all $A \in \mathfrak{h}$ are skew-adjoint relative to some pseudo-Euclidean inner product in V.

Given a manifold M, consider the infinite-dimensional affine space $\mathcal{C}(M)$ of all torsion-free connections on M and, for any fixed differential p-form μ on M,

(12.3) the affine mapping $\mathcal{C}(M) \ni \nabla \mapsto \nabla\mu$,

valued in $(0, p+1)$ tensor fields, skew-symmetric in the last p arguments.

LEMMA 12.1. *Let an algebraically constant differential* 3*-form* μ *on an* n*-dimensional manifold* M *have an algebraic type that satisfies* (12.2). *Then the mapping* (12.3) *is injective and, even locally, there exists at most one torsion-free connection* ∇ *on* M *with* $\nabla\mu = 0$.

PROOF. Suppose that a $(1,2)$ tensor field B is the difference between two torsion-free connections on M assigining to μ the same covariant derivative. Thus, in local coordinates, $B^s_{ij}\mu_{skq} + B^s_{ik}\mu_{jsq} + B^s_{iq}\mu_{jks} = 0$, that is, if a vector v is tangent to M at a point x, the $(1,1)$ tensor $B_x(v, \cdot\,)$ equals, by (12.1), the value at x of some element A of the isotropy Lie algebra $\mathfrak{h} \subseteq \mathfrak{gl}(T_xM)$ of μ_x. For a pseudo-Euclidean inner product g in T_xM chosen as in (12.2), $g_{ks}B^s_{ij}(x)$ is thus symmetric in i, j and skew-symmetric in j, k, so that it must vanish, proving the injectivity claim. The remaining assertion is now obvious, with the 'even locally' part immediate as M may be replaced by any open submanifold. □

Following Joyce [**12**, Sect. 2.2–2.3], for a Euclidean vector space V of dimension n and a basis $\xi^1, \dots, \xi^n$ of V^* dual to an orthonormal basis of V, we write $\xi^{ij\dots k}$ for $\xi^i \wedge \xi^j \wedge \dots \wedge \xi^k$, and consider, when $n = 7$, the exterior 3-form

$$\mu = \xi^{123} + \xi^{145} + \xi^{167} + \xi^{246} - \xi^{257} - \xi^{347} - \xi^{356}, \tag{12.4}$$

while, if $n = 8$, we use the same symbol for the exterior 4-form

$$\begin{aligned} \mu = {} & \xi^{1234} + \xi^{1256} + \xi^{1278} + \xi^{1357} - \xi^{1368} - \xi^{1458} - \xi^{1467} \\ - {} & \xi^{2358} - \xi^{2367} - \xi^{2457} + \xi^{2468} + \xi^{3456} + \xi^{3478} + \xi^{5678}. \end{aligned} \tag{12.5}$$

In both cases, the isotropy group of μ in $\mathrm{GL}(V)$, isomorphic to G_2 or, respectively, $\mathrm{Spin}(7)$, preserves the inner product [**12**, Sect. 2.2–2.3]. Thus,

(12.6) both (12.4) and (12.5) have the property (12.2).

13. The Cartan 3-forms of simple Lie algebras

Our convention about the sign of the curvature tensor R of a connection ∇ on a manifold M is such that, for vector fields u, v, w,

$$R(v,w)u = \nabla_w \nabla_v u - \nabla_v \nabla_w u + \nabla_{[v,w]} u \tag{13.1}$$

When ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g on M, we may treat R as a vector-bundle morphism

(13.2) a) $R : [T^*M]^{\otimes 2} \to [T^*M]^{\otimes 2}$, b) leaving $[T^*M]^{\odot 2}$ and $[T^*M]^{\wedge 2}$ invariant,

so that it acts on arbitrary $(0,2)$ tensor fields b by

$$[Rb]_{ij} = R_{ipjq} b^{pq}, \tag{13.3}$$

with index raising and lowering via g, and summation over repeated indices. See [**2**, Sect. 1.114, 1.131]. This action not only preserves (skew)symmetry of b, but also – due to the first Bianchi identity – amounts to the usual formula

$$2[R\zeta]_{ij} = R_{ijpq} \zeta^{pq} \tag{13.4}$$

if $b = \zeta$ happens to be skew-symmetric (a 2-form).

Given a connected Lie group G, with the Lie algebra $\mathfrak{g}$ of left-invariant vector fields, we treat the Killing form g, the Cartan 3-form γ, characterized by $g(u,v) = \mathrm{tr}\,[(\mathrm{Ad}\,u)\mathrm{Ad}\,v]$ and $\gamma(u,v,w) = g([u,v],w)$ whenever $u, v, w \in \mathfrak{g}$, and the Lie bracket $C : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, as left-invariant tensor fields of types $(0,2)$, $(0,3)$ and $(1,2)$ on G, which then makes them – see below – also bi-invariant. Setting

$$\nabla_v w = [v,w]/2 \quad \text{for} \quad v, w \in \mathfrak{g}, \tag{13.5}$$

we define the *standard bi-invariant torsion-free connection* ∇ on G. By (13.1), ∇ has the ∇-parallel curvature tensor R with

$$4R(v,w)u = [[v,w],u] \text{ for } u, v, w \in \mathfrak{g}. \tag{13.6}$$

Bi-invariance of C and ∇ trivially follows from the diffeomorphic invariance of the Lie bracket, while that of g (and, consequently, γ), as well as the fact that g, γ, C and R are all ∇-parallel, is due to the Jacobi identity.

In any local coordinates $x^1, \ldots, x^n$ for G, unrelated to the Lie-group structure, our tensor fields have the component functions g_{ij}, γ_{ijk} and C_{ij}^k, with

$$\text{(13.7)} \qquad \text{a) } g_{ij} = C_{ir}^s C_{js}^r, \qquad \text{b) } \gamma_{ijk} = C_{ij}^r g_{rk}, \qquad \text{c) } 4R_{ijk}{}^q = C_{ij}^r C_{rk}^q.$$

In the case where $\mathfrak{g}$ is semisimple, which allows us to use g-index raising, g is clearly a locally symmetric pseudo-Riemannian Einstein metric with the Levi-Civita connection ∇, while (13.7-a) and (13.7-c) can be rewritten as

$$\text{(13.8)} \qquad \gamma_{ipq}\gamma^{pqj} = -\delta_i^j, \qquad 4R_{ijkq} = C_{ij}^r \gamma_{rkq} = g^{rs}\gamma_{ijr}\gamma_{kqs}.$$

The dimension restriction in the next lemma amount to requiring that $\mathfrak{g}$ not be isomorphic to $\mathfrak{sl}(2,\mathrm{I\!R})$, $\mathfrak{sl}(2,\mathbb{C})$, or $\mathfrak{su}(2) = \mathfrak{so}(3)$.

Lemma 13.1. *For any simple Lie algebra* $\mathfrak{g}$ *of real dimension* $n \geq 8$*, the isotropy Lie algebra* $\mathfrak{h} \subseteq \mathfrak{gl}(\mathfrak{g})$*, with* (12.1)*, of the Cartan* 3*-form* $\gamma \in [\mathfrak{g}^*]^{\wedge 3}$ *equals the image* $\{\mathrm{Ad}\, v : v \in \mathfrak{g}\}$ *of the* Ad *representation.*

Consequently, (12.2) *holds for* $V = \mathfrak{g}$ *and* $\mu = \gamma$*, and on any connected simple Lie group* G *of dimension* $n \geq 8$ *the standard bi-invariant torsion-free connection* ∇ *given by* (13.5) *is, even locally, the only torsion-free connection on* G *that makes the Cartan* 3*-form* γ *parallel.*

Proof. Let us fix $A \in \mathfrak{h}$ and identify $\mathfrak{g}$ with the Lie algebra of left-invariant vector fields on a connected Lie group G. Then, in local coordinates $x^1, \ldots, x^n$ as above, A treated as a left-invariant tensor field of type $(1,1)$ on G satisfies the relation $A_i^s\gamma_{sjk} + A_j^s\gamma_{isk} + A_k^s\gamma_{ijs} = 0$ which, contracted against γ^{jkp}, gives, due to (13.8), $A_j^i = 8A_p^q R^{ip}{}_{jq}$. In other words, $a^* = 8Ra$ for the $(0,2)$ tensors a, a^* at any point $x \in G$ defined by $a_{ij} = A_i^s g_{sj}$ and $a_{ij}^* = a_{ji}$. The operator (13.2-a) for R in (13.6) and $M = G$ commutes, by (13.2-b), with $a \mapsto a^*$, and so

$$\text{(13.9)} \qquad 8Ra^{\pm} = \pm a^{\pm} \;\text{ for }\; a^{\pm} = (a \pm a^*)/2.$$

The spectrum of (13.2-a) for R in (13.6) is completely understood for all simple Lie groups, via an easy argument in [**7**] for the restriction $R : [\mathfrak{g}^*]^{\wedge 2} \to [\mathfrak{g}^*]^{\wedge 2}$, valid in the general semisimple case, and, for the other restriction, $R : [\mathfrak{g}^*]^{\odot 2} \to [\mathfrak{g}^*]^{\odot 2}$, due to a result of Meyberg [**14**], also presented in [**7**, the Appendix].

Namely, the operator T in [**7**, formula (2.6)] is, by (13.3) and (13.8), equal to $-8R$, for our R in (13.2-a) and (13.6), and so, according to [**7**, Lemma 2.1(c)–(d)], $R : [\mathfrak{g}^*]^{\wedge 2} \to [\mathfrak{g}^*]^{\wedge 2}$ is diagonalizable with the eigenvalues 0 and $-1/8$, while its eigenspace for the eigenvalue $-1/8$ is $\{\gamma(v, \cdot\,, \cdot\,) : v \in \mathfrak{g}\}$.

On the other hand, under our assumption about the dimension of $\mathfrak{g}$, [**7**, Remark 4.5] implies that $1/8$ is *not* an eigenvalue of $R : [\mathfrak{g}^*]^{\odot 2} \to [\mathfrak{g}^*]^{\odot 2}$. Note that, according to [**7**, Lemma 2.1(b)], $\mathit{\Omega}$ in [**7**, Remark 4.5] equals T, and hence our $-8R$. Thus, by (13.9), $a = a^-$ lies, at each point, in $\{\gamma(v, \cdot\,, \cdot\,) : v \in \mathfrak{g}\}$, which is the eigenspace just mentioned, that is, $A \in \{\mathrm{Ad}\, v : v \in \mathfrak{g}\}$. As the opposite inclusion $\{\mathrm{Ad}\, v : v \in \mathfrak{g}\} \subseteq \mathfrak{h}$ amounts to the aforementioned bi-invariance of γ, the first part of the lemma follows, while the final clause is then obvious from Lemma 12.1, since $\nabla\gamma = 0$ the according to the lines following (13.6). □

14. Parallelism without local constancy

As mentioned in the Introduction, the converse of the first implication in (1.2) for p-forms in dimension n fails in general, unless $p \in \{0, 1, 2, n-2, n-1, n\}$. Here are some explicit examples.

THEOREM 14.1. *If an exterior p-form in dimension n has the algebraic type*

(a) *of* (12.4) *with* $(n,p)=(7,3)$, *or* (12.5) *for* $(n,p)=(8,4)$, *or*

(b) *the Cartan* 3*-form of any simple Lie algebra* $\mathfrak{g}$ *with* $\dim_{\mathrm{I\!R}}\mathfrak{g}=n\geq 8$,

then it can be realized as a differential p-form μ on an n-dimensional manifold, so as to be parallel, but not locally constant on any open submanifold.

Specifically, for (a) *we may choose μ to be a specific parallel p-form on a compact simply connected Riemannian manifold of dimension $n\in\{7,8\}$ with the holonomy group G_2 or* Spin(7) *while, for* (b), *we let μ be the Cartan* 3*-form on a connected Lie group having $\mathfrak{g}$ as the Lie algebra of left-invariant vector fields.*

PROOF. Our choice for (a) is possible according to Joyce [**12**, Sect. 2.2–2.3], with the resulting Levi-Civita connection ∇ that must be Ricci-flat, and hence real-analytic [**9**]. By (12.6) and Lemma 12.1, ∇ is, even locally, the only torsion-free connection with $\nabla\mu=0$. Having the holonomy group G_2 or Spin(7), it is not flat either on M, or – due to analyticity – on any open submanifold.

For (b), our assertion is in turn obvious from the uniqueness assertion in the final clause of Lemma 13.1 combined with (1.1), since ∇ in Lemma 13.1 is – according to the lines preceding (13.8) – the Levi-Civita connection of the pseudo-Riemannian Einstein metric g (the Killing form), which is not flat on any open submanifold, as it has, by (13.8), the nonzero parallel Ricci tensor $-g/4$. □

REMARK 14.2. Any real simple Lie algebra $\mathfrak{g}$ is either a real form of a complex simple Lie algebra $\mathfrak{h}$, or the result of treating some such $\mathfrak{h}$ as real. See, e.g., [**10**, Lemma 4 on p. 173]. According to [**7**, Theorem 4.1], the curvature operator $R:[\mathfrak{g}^*]^{\odot 2}\to[\mathfrak{g}^*]^{\odot 2}$ in Sect. 13 has the same nonzero eigenvalues as its analog for $\mathfrak{h}$. It also behaves additively under the direct-sum operation applied to Lie algebras. This generalizes the final clause of Lemma 13.1 and Theorem 14.1(b) to the case of arbitrary semisimple Lie algebras without ideals of dimensions 3 or 6.

15. Logical independence in Theorems 9.2 and 9.3

Closedness of an algebraically constant differential p-form μ in dimension n is known *not* to imply integrability of its divisibility distribution $\mathcal{D}$ except when $p\in\{0,1,2,n-1,n\}$: counterexamples in [**8**, Sect. 12] realize all dimensions $n\geq 5$ and all p with $2<p<n-1$.

It is thus natural to ask if other parts of items (iii) in Theorem 9.2 and (i), (ii) in Theorem 9.3 are similarly free of redundancy. Theorem 9.2 gives rise to three questions of this kind – whether closedness of μ implies any of (a), (b), (c) – and Theorem 9.3 to one more: does (ii) follow from (i)?

This section answers the first three questions in the negative with the aid of the following examples. They use $e_1,\dots,e_6$, dual to $\xi^1,\dots,\xi^6$ as in (8.1), chosen to form a basis of the Lie algebra of left-invariant vector fields on a Lie group, the only nonzero Lie brackets being those algebraically related to

$$\begin{array}{lll}[e_1,e_2]=e_5, & [e_1,e_3]=e_6 & \text{for (a)},\\ [e_2,e_4]=e_1, & [e_6,e_2]=e_3 & \text{for (b)},\\ [e_1,e_2]=e_6, & [e_5,e_4]=e_3 & \text{for (c)}.\end{array}\tag{15.1}$$

The Jacobi identity is obvious, since all brackets lie in the center. The only nonzero components of μ thus are, cf. (6.1), up to obvious consequences of skew-symmetry, $\mu_{123}=\mu_{345}=\mu_{561}=\mu_{246}=1$ for (a), $\mu_{123}=\mu_{345}=\mu_{561}=1$ for (b),

$\mu_{123} = \mu_{456} = 1$ for (c). By (2.2), with the same convention, the only possibly-nonzero components of $\zeta = d\mu$ are ζ_{ijkl} such that $[e_i, e_j] \neq 0$, Explicitly, in case (a): $\zeta_{12kl} = 0$ and $\zeta_{13ij} = 0$ for kl, or ij, ranging over $34, 35, 36, 45, 46, 56$ or, respectively, $45, 46, 56$. Similarly, for (b): $\zeta_{24kl} = 0$ with $kl = 13, 15, 16, 35, 36, 56$, and $\zeta_{26ij} = 0$ for $ij = 13, 15, 35$. Finally, in case (c), $\zeta_{12kl} = 0$ and $\zeta_{45ij} = 0$ with kl, or ij, ranging over $34, 35, 36, 45, 46, 56$ or, respectively, $13, 16, 23, 26, 36$. Thus, $d\mu = 0$ in all cases, while (15.1) and (8.3) show that neither $\mathcal{H}$, for (b), nor either of $\mathcal{H}^{\pm}$, for (c), is integrable. Finally, the Nijenhuis tensor N of J sends vector fields v, w to $N(v, w) = J[Jv, w] + J[v, Jw] - [Jv, Jw] + [v, w]$, so that $N(e_1, e_2) = e_5$ by (15.1) and (6.4), proving non-integrability of J in case (a).

For the remaining (fourth) redundancy question raised in the initial paragraph of this section, a negative answer is provided by Remark 16.3 below.

16. Duality and closedness

Given a nondegenerate 2-form σ on an n-dimensional manifold M, with $n = 2m \geq 2$ even, we use (5.7) to define its *dual* $(n-2)$*-form* μ. Note that

(16.1) if σ is closed, or locally constant, or parallel, then so is, respectively, its dual μ,

the locally-constant and parallel cases obvious as μ and σ arise from each other via explicit constructions (Lemma 5.2). For the same reason, conversely,

(16.2) parallelism or local constancy of the dual of σ implies the same for σ.

Now (16.1) follows: closedness of σ is equivalent to its local constancy due to the Darboux theorem, which gives $\sigma = dx^1 \wedge dx^2 + \ldots + dx^{n-1} \wedge dx^n$ in suitable local coordinates $x^1, \ldots, x^n$. Thus, by (5.8), the dual μ of σ has constant component functions in these coordinates, and is consequently closed.

The symbol $\widehat{\ }$ means 'delete' in the following theorem, which shows that, in contrast with (16.2), the converse of the "closed" case of (16.1) is generally false in all (necessarily even) dimensions $n \geq 6$. Note that for $n = 2$ and $n = 4$ the converse is true, since then $\mu = -1$ and $\mu = -\sigma$ by (5.9).

The index ranges below are $i, j = 1, \ldots, m$ and $k = 1, \ldots, 2m$.

THEOREM 16.1. *Given an integer $m \geq 2$ and positive functions ϕ_i of the $2m$ real variables x^k, let $\chi_i = dx^{2i-1} \wedge dx^{2i}$ and $\sigma_i = \phi_i^{1-m} \phi \chi_i$, where $\phi = \phi_1 \ldots \phi_m$. The 2-form σ and $(2m-2)$-form ζ defined by $\sigma = \sigma_1 + \ldots + \sigma_m$ and $\zeta = \zeta_1 + \ldots + \zeta_m$, with $\zeta_i = -\sigma_1 \wedge \ldots \wedge \widehat{\sigma}_i \wedge \ldots \wedge \sigma_m$, then have the following properties.*

- (a) *ζ is algebraically constant and indivisible.*
- (b) *σ and ζ are dual to each other.*
- (c) *ζ is closed if and only if, with no summation, $\partial_{2i-1}\phi_i = \partial_{2i}\phi_i = 0$.*
- (d) *σ is closed if and only if $\partial_k[\phi_i^{1-m}\phi] = 0$ whenever $k \notin \{2i-1, 2i\}$, which is in turn equivalent to having $\phi_i = \rho_1 \ldots \widehat{\rho}_i \ldots \rho_m$ for all i, where each ρ_j is a function of x^{2j-1} and x^{2j}.*

PROOF. Algebraic constancy of ζ follows as $\sigma_i = \xi^{2i-1} \wedge \xi^{2i}$ with $\xi^k = [\phi_i^{1-m}\phi]^{1/2} dx^k$ for $k \in \{2i-1, 2i\}$. Now (5.8) implies (b), and hence (a).

Next, for $\theta_k = dx^1 \wedge \ldots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \ldots \wedge dx^{2m}$, the obvious relation $\zeta_i = -\phi_i^{m-1} \chi_1 \wedge \ldots \wedge \widehat{\chi}_i \wedge \ldots \wedge \chi_m$ implies that $d\zeta_i$ equals the combination of θ_{2i} and θ_{2i-1} with the coefficients $-\partial_{2i-1}[\phi_i^{m-1}]$ and $-\partial_{2i}[\phi_i^{m-1}]$. As $\zeta = \zeta_1 + \ldots + \zeta_m$, linear independence of the $(2m-1)$-forms $\theta_1, \dots, \theta_{2m}$ yields (c). Similarly, $d\sigma_i$ is the combination of $dx^k \wedge dx^{2i-1} \wedge dx^{2i}$, over $k \notin \{2i-1, 2i\}$, with the coefficients $\partial_k[\phi_i^{1-m}\phi]$. Linear independence of all such $dx^k \wedge dx^{2i-1} \wedge dx^{2i}$ now proves the first claim in (d). Since, for purely algebraic reasons, $\phi_i = \rho_1 \dots \widehat{\rho}_i \dots \rho_m$ for all i if and only if $\phi_i^{1-m}\phi = \rho_i^{m-1}$, the second part of (d) follows. □

REMARK 16.2. Starting from $m = 3$, Theorem 16.1 yields examples in which ζ is closed, but its dual σ is not: we may clearly choose ϕ_i as in (c) that do not have separated variables in the sense of (d).

REMARK 16.3. For a nonzero differential $(n-2)$-form on an n-dimensional manifold M, satisfying (7.1-b) and having the divisibility distribution $\mathcal{D}$ of fibre dimension q, so that q is even and $2 \le q \le n$ due to (7.3), closedness of μ and integrability of $\mathcal{D}$ do not imply closedness, along the leaves of $\mathcal{D}$, of the 2-form σ dual to the indivisible factor ζ of μ. Examples realizing all n, q with even q and $n \ge q \ge 6$ arise from Remark 7.3 applied to a manifold Σ of any dimension $s \ge 0$ and an open submanifold Π of $\mathbb{R}^{2m}$, $m \ge 3$, with the $(2m-2)$-form ζ of Theorem 16.1 chosen so as to be closed without closedness of its dual 2-form σ (see Remark 16.2). Here $n = 2m + s$ and $q = 2m$.

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