The curvature of left-invariant magnetic systems

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Abstract. We compute the magnetic curvatures (in the sense of Assenza) of left-invariant magnetic systems on Lie groups and explore relations between curvature properties and algebraic properties, extending to the magnetic setting a number of results from Milnor's classic 1976 paper on the geometry of left-invariant metrics. We also discuss the existence of non-trivial bi-invariant magnetic systems and exhibit the resulting curvature formulas in some concrete examples, including the Heisenberg group.

1. Introduction

A *magnetic system* on a smooth manifold M is defined, as usual, to be a pair (g,σ), where g is a Riemannian metric and σ is a closed 2-form on M; in this context, σ is called the *magnetic 2-form*. One may then consider the second-order ordinary differential equation

$$\frac{\mathrm{D}\dot\gamma}{\mathrm{d}t}(t) = \mathbf{Y}_{\gamma(t)}(\dot\gamma(t)), \tag{1.1}$$

for smooth curves $\gamma\colon I \to M$, where the covariant derivative $\mathrm{D}/\mathrm{d}t$ is induced by the Levi-Civita connection of g, and the *Lorentz force operator* $\mathbf{Y}\colon TM \to TM$, skew-adjoint and equivalent to σ under g, is characterized by

$$\sigma_x(v,w) = \mathrm{g}_x(\mathbf{Y}_x(v),w), \quad \text{for all } x \in M \text{ and } v,w \in T_xM. \tag{1.2}$$

Equation (1.1) is sometimes called the *Landau-Hall equation*, and its solutions are called (g,σ)-geodesics; they all have constant speed due to skew-adjointness of $\mathbf{Y}$, and (g,σ) is called *complete* when all inextendible (g,σ)-geodesics are defined on all of $\mathbb{R}$. In this case, the magnetic flow obtained from (1.1) may be restricted to each energy level $\Sigma_s = \{(x,v) \in TM : \|v\|_x = s\}$, where $s > 0$. The dynamical and geometric properties of such magnetic flows, that now strongly depend on the

2020 *Mathematics Subject Classification.* Primary: 53C15. Secondary: 22E15 · 53C30.

Key words and phrases. magnetic systems · magnetic curvature · Lie groups.

value of s (unlike in the classical Riemannian case, where $\sigma = 0$), have been—and continue to be—largely investigated. See for instance [**12**, **19**, **26**, **35**, **41**] and the references therein, or [**1**, **2**, **6**, **15**, **33**] for more recent work.

With the recent introduction by Assenza [**7**] of a *magnetic curvature operator*, and the companion notions of *magnetic sectional and Ricci curvatures*, many results in Riemannian geometry have been generalized to the setting of magnetic systems. For example, we have the Bonnet-Myers and Synge's theorems in [**7**], Green and Hopf's theorems in [**9**], compatibility equations from submanifold theory have been listed in [**39**], and an interpretation of magnetic curvature as the curvature of a certain Cartan connection is given in the recent preprint [**14**]. We review the precise definitions of magnetic curvature, all of which explicitly carry the parameter $s > 0$ from the previous paragraph, in Section 2.

Here, we consider the simplest prototypes of homogeneous spaces: Lie groups. Naturally, a magnetic system on a Lie group is *left-invariant* if all left-translations are *magnetomorphisms* (that is, diffeomorphisms preserving both the metric and magnetic 2-form, cf. [**2**]). Explicit descriptions of magnetic geodesics have been obtained in Heisenberg groups [**25**, **37**], and in three-dimensional groups with bi-invariant metrics [**40**], while integrability of magnetic flows (both in the Liouville and non-commutative senses) on Lie groups and homogeneous spaces has been thoroughly studied in [**17**,**23**,**24**,**32**]. Our goal here is to study curvature. Following in spirit parts of the presentation by Milnor [**36**], we register formulas for the magnetic curvatures of left-invariant magnetic systems in Section 4, and discuss some relations between curvature properties and algebraic ones.

Our main results follow the general trend of many rigidity-related theorems for magnetic systems, where the magnetic 2-form ultimately vanishes; see e.g. [**21**, Lemma A.7], [**8**, Corollary 1.6], [**34**, Theorem A and Corollary 3.6], and [**9**, Theorem D]. The first one we generalize is a well-known consequence of [**36**, formula (7.3)], about the sectional curvature of a bi-invariant metric:

Theorem A. *For any Lie group G and bi-invariant magnetic system (g, σ) on G, and any $s > 0$, it holds that $\mathrm{Sec}^{(\mathrm{g},\sigma,s)} \geq 0$. Moreover, $\mathrm{Sec}^{(\mathrm{g},\sigma,s)} = 0$ if and only if G is Abelian and $\sigma = 0$.*

As in [**36**], Theorem A follows from a more general computation—summarized in Proposition 4.3—of the s-magnetic sectional curvature along directions which are "compatible" with (g, σ), in the sense of Proposition 4.1.

Next, we have a magnetic version of [**36**, Theorem 1.5], which specializes [**9**, Theorem D] to the left-invariant setting:

THEOREM B. *Let* (g, σ) *be a left-invariant magnetic system on a connected Lie group* G. *Then, there is* $s > 0$ *such that* $\mathrm{Sec}^{(\mathrm{g},\sigma,s)} = 0$ *if and only if* $\nabla\sigma = 0$ *and one of the following occurs:*

(i) $\sigma = 0$ *and the Lie algebra of* G *splits as an orthogonal direct sum* $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$, *where* $\mathfrak{b}$ *is a commutative subalgebra,* $\mathfrak{u}$ *is a commutative ideal, and* ad_b *is skew-adjoint for every* $b \in \mathfrak{b}$.
(ii) σ *is nowhere-vanishing and* $J = \|\mathbf{Y}\|^{-1}\mathbf{Y}$ *makes* (G, g) *a Kähler manifold holomorphically isometric to a complex hyperbolic space* $\mathbb{CH}^n$.

The main ingredient in the proof of Theorem B, perhaps hinted at by item (ii) therein, is a classical result by Heintze: every connected homogeneous Kähler manifold of negative curvature is holomorphically isometric to a complex hyperbolic space [**27**, Theorem 4].

Finally, we also obtain a magnetic version of [**36**, Theorem 1.6]:

THEOREM C. *Let* G *be a connected Lie group, and* (g, σ) *be a left-invariant magnetic system on* G. *If there is* $s > 0$ *such that* $\mathrm{Sec}^{(\mathrm{g},\sigma,s)} = 0$, *then* G *is solvable. If, in addition,* G *is unimodular, then* $\sigma = 0$ *and* g *is flat.*

The main assumption in Theorem C is much more restrictive than the one in [**36**, Theorem 1.6], which allows the sectional curvature to be nonpositive. We conjecture, however, that Theorem C remains true if one instead assumes that $\mathrm{Sec}^{(\mathrm{g},\sigma,s)} \leq 0$. The difficulties with a possible proof seem to ultimately stem from the general failure of the Hopf-Rinow theorem in the magnetic setting [**13**, Remark 2.5(b)], causing a crucial step in the proof of the corresponding would-be magnetic Cartan-Hadamard theorem to fail. Without it, it is not clear how to trace out the extensive work of Wolf [**42**], Azencott and Wilson [**10**,**11**], Kobayashi [**29**], and others in order to obtain a classification of negatively curved homogeneous magnetic systems.

Organization of the paper. Throughout the entire text, unless otherwise stated, we work in the smooth category, G denotes a connected Lie group, and $\mathfrak{g}$ is its Lie algebra. In Sections 2 and 3 we review basic facts about magnetic curvature and left-invariant metrics. In Section 4 we put such facts together and register in Proposition 4.2 an explicit formula for the s-magnetic sectional curvature of left-invariant magnetic systems; a few consequences—including Theorem A—are then obtained. Next, in Section 5 we discuss the existence of nontrivial bi-invariant magnetic systems, and in Section 6 we prove Theorems B and C. Finally, in Section 7 we compute the magnetic curvatures of a special class of systems on Heisenberg groups. Appendix A derives a generalization of the well-known Euler-Arnold formalism, for left-invariant exact magnetic systems.

2. Magnetic curvature

We briefly review, for the reader's convenience, the definitions of magnetic curvature presented by Assenza [**7**]. See also [**9**, pp. 2–3], [**34**, p. 2], and [**39**, p. 4].

Let (g,σ) be a magnetic system on a smooth manifold M, and let $UM \to M$ be the unit-tangent bundle of (M,g). We consider two auxiliary bundles: the Stiefel bundle $\mathrm{St}_2(M,\mathrm{g}) \to M$ of orthonormal 2-frames tangent to M, and the bundle $E \to UM$ of orthogonal hyperplanes tangent to M; specifically, their fibers are given by $\mathrm{St}_2(M,\mathrm{g})_x = \{(v,w) \in (U_xM)^{\times 2} : \mathrm{g}_x(v,w) = 0\}$ and $E_{(x,v)} = v^\perp$.

For each $s > 0$, the *s-magnetic curvature operator* of (g,σ) is the self-adjoint endomorphism $\boldsymbol{M}^{(\mathrm{g},\sigma,s)} \colon E \to E$ given by

$$\boldsymbol{M}^{(\mathrm{g},\sigma,s)}_{(x,v)}(w) = R^{(\mathrm{g},\sigma,s)}_{(x,v)}(w) + A^{(\mathrm{g},\sigma)}_{(x,v)}(w), \tag{2.1}$$

where $R^{(\mathrm{g},\sigma,s)}, A^{(\mathrm{g},\sigma)} \colon E \to E$ are in turn defined as

$$\begin{aligned} A^{(\mathrm{g},\sigma)}_{(x,v)}(w) &= -\frac{3}{4}\mathbf{Y}_x(\mathbf{Y}_x(w)^\top) - \frac{1}{4}\mathbf{Y}_x^2(w)^\perp, \quad \text{and} \\ R^{(\mathrm{g},\sigma,s)}_{(x,v)}(w) &= s^2 R^{\mathrm{g}}_x(w,v)v - s(\nabla_w \mathbf{Y})(v) + \frac{s}{2}(\nabla_v \mathbf{Y})(w)^\perp, \end{aligned} \tag{2.2}$$

where R^{g} is the Riemann curvature tensor of g, and the tangent and perpendicular projections are taken relative to the orthogonal splitting $T_xM = \mathbb{R}v \oplus v^\perp$ induced by g. The general formulas

$$\begin{aligned} &\mathrm{g}_x(A^{(\mathrm{g},\sigma)}_{(x,v)}(w), w) = \frac{3}{4}\mathrm{g}_x(\mathbf{Y}_x(v), w)^2 + \frac{1}{4}\|\mathbf{Y}_x(w)\|^2 \\ &\text{and} \quad \mathrm{tr}\big(A^{(\mathrm{g},\sigma)}_{(x,v)}\big) = \frac{1}{2}\|\mathbf{Y}_x(v)\|^2 + \frac{1}{4}\|\mathbf{Y}_x\|^2, \quad \text{where } \|\mathbf{Y}_x\| = \sqrt{-\mathrm{tr}(\mathbf{Y}_x^2)}, \end{aligned} \tag{2.3}$$

clear from (2.2), will be useful in Sections 4 and 7. The *s-magnetic sectional curvature* $\mathrm{Sec}^{(\mathrm{g},\sigma,s)} \colon \mathrm{St}_2(M,\mathrm{g}) \to \mathbb{R}$ and *s-magnetic Ricci curvature* $\mathrm{Ric}^{(\mathrm{g},\sigma,s)} \colon UM \to \mathbb{R}$ are defined by

$$\mathrm{Sec}^{(\mathrm{g},\sigma,s)}_x(v,w) = \mathrm{g}_x\big(\boldsymbol{M}^{(\mathrm{g},\sigma,s)}_{(x,v)}(w), w\big) \quad \text{and} \quad \mathrm{Ric}^{(\mathrm{g},\sigma,s)}(x,v) = \mathrm{tr}\,\big(\boldsymbol{M}^{(\mathrm{g},\sigma,s)}_{(x,v)}\big). \tag{2.4}$$

The s-magnetic sectional curvature does not survive as a function on the Grassmannian bundle of 2-planes tangent to M with,

$$\text{in general, } \mathrm{Sec}^{(\mathrm{g},\sigma,s)}_x(v,w) \neq \mathrm{Sec}^{(\mathrm{g},\sigma,s)}_x(w,v). \tag{2.5}$$

As in the Riemannian case, however, the s-magnetic Ricci curvature may be expressed in terms of the s-magnetic sectional curvature, with

$$\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(x,v) = \sum_{i=1}^{n-1} \mathrm{Sec}_x^{(\mathrm{g},\sigma,s)}(v,e_i), \tag{2.6}$$

where $e_1,\dots,e_{n-1}$ is any orthonormal basis of $v^\perp$. Some useful relations between the magnetic curvatures and the Riemannian ones are also readily obtained, e.g.

$$\mathrm{Sec}_x^{(\mathrm{g},\sigma,s)}(v,w) = s^2\mathrm{Sec}_x^{\mathrm{g}}(v,w) - s\mathrm{g}_x((\nabla_w \mathbf{Y})(v),w) + \mathrm{g}_x\big((A_{(x,v)}^{(\mathrm{g},\sigma)}(w),w\big), \tag{2.7}$$

cf. [**9**, Formula (6.3)], and from there with (2.6),

$$\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(x,v) = s^2\mathrm{Ric}_x^{\mathrm{g}}(v,v) - s\,(\mathrm{div}_{\mathrm{g}}\sigma)_x(v) + \mathrm{tr}\big(A_{(x,v)}^{(\mathrm{g},\sigma)}\big), \tag{2.8}$$

where $\mathrm{div}_{\mathrm{g}}\sigma \in \Omega^1(M)$ is given by $(\mathrm{div}_{\mathrm{g}}\sigma)(Y) = \mathrm{tr}_{\mathrm{g}}((X,Z)\mapsto(\nabla_X\sigma)(Y,Z))$.

3. Riemannian curvature of Lie groups

We present, for later reference and to establish notation, some general facts about left-invariant metrics on Lie groups. In particular, we refer to [**20**, Proposition 3.18], [**16**, Theorem 7.30], or [**5**, Theorem 5.3] for more details on the formulas involving curvature.

Let G be a connected Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathrm{g} = \langle\cdot,\cdot\rangle$ be a left-invariant metric on G. For all $x,y\in\mathfrak{g}$, we set $\mathrm{ad}_x y = [x,y]$, and denote by $\mathrm{ad}_x^\dagger$ the g-adjoint of ad_x, characterized by $\langle \mathrm{ad}_x y, z\rangle = \langle y, \mathrm{ad}_x^\dagger z\rangle$ for all $z\in\mathfrak{g}$. The Levi-Civita connection of g is given by the Koszul formula

$$\nabla_x y = \frac{1}{2}[x,y] - B(x,y), \quad \text{where} \quad B(x,y) = \frac{1}{2}(\mathrm{ad}_x^\dagger y + \mathrm{ad}_y^\dagger x). \tag{3.1}$$

We also have that g is bi-invariant if and only if $B=0$, as a consequence of the easily established formula $-2\langle B(x,y),z\rangle = \langle x, \mathrm{ad}_z y\rangle + \langle \mathrm{ad}_z x, y\rangle$. Indeed, g is bi-invariant if and only if each ad_z is a skew-adjoint operator [**36**, Lemma 7.2]. Moreover, it directly follows from (3.1) that, whenever $z,v\in\mathfrak{g}$ and ad_z is skew-adjoint, the relations

$$\text{(i) } \nabla_z z = 0, \quad \text{(ii) } \nabla_v z = -\frac{1}{2}\mathrm{ad}_v^\dagger z, \quad \text{(iii) } \nabla_z v = \mathrm{ad}_z v - \frac{1}{2}\mathrm{ad}_v^\dagger z \tag{3.2}$$

all hold. Finally, we have that

$$\begin{aligned}\mathrm{Sec}^{\mathrm{g}}(v,w) = &-\frac{3}{4}\|[v,w]\|^2 - \frac{1}{2}\langle[[v,w],w],v\rangle \\ &-\frac{1}{2}\langle[v,[v,w]],w\rangle - \langle B(v,v),B(w,w)\rangle + \|B(v,w)\|^2\end{aligned} \tag{3.3}$$

whenever $v,w\in\mathfrak{g}$ are orthonormal vectors; in particular, whenever $B=0$, (3.3) reduces to $\mathrm{Sec}^{\mathrm{g}}(v,w) = \|[v,w]\|^2/4$.

4. Left-invariant magnetic systems and sectional curvature

Let G be a connected Lie group, and $\mathfrak{g}$ its Lie algebra. A left-invariant 2-form σ on G is closed if and only if

$$\sigma([u,v],w)+\sigma([v,w],u)+\sigma([w,u],v)=0,\quad \text{for all } u,v,w\in\mathfrak{g}. \tag{4.1}$$

If σ is exact, and G is compact or has $H^2(\mathfrak{g},\mathbb{R})=\{0\}$, we may choose a left-invariant 1-form α on G such that $\sigma=-\mathrm{d}\alpha$, in which case $\sigma(u,v)=\alpha([u,v])$ for all $u,v\in\mathfrak{g}$. If $\mathrm{g}=\langle\cdot,\cdot\rangle$ is a left-invariant metric on G, we will often start with a skew-adjoint operator $\mathbf{Y}$ on $\mathfrak{g}$, and *define* σ via (1.2) instead; it follows from (1.2) and (4.1) that

$$\text{if } [\mathfrak{g},\mathfrak{g}]\subseteq\ker\mathbf{Y}, \text{ the corresponding 2-form } \sigma \text{ is closed.} \tag{4.2}$$

Next, a left-invariant covariant tensor field Θ is bi-invariant if and only if it is Ad-invariant. This latter condition, as G is connected, may be tested for infinitesimally: it holds if and only if

$$\sum_{i=1}^{k}\Theta(v_1,\ldots,v_{i-1},\mathrm{ad}_z v_i,v_{i+1},\ldots,v_k)=0, \tag{4.3}$$

for all $z,v_1,\ldots,v_k\in\mathfrak{g}$. It now follows that

$$\text{if the metric g is bi-invariant, } \Theta \text{ is bi-invariant if and only if } \Theta \text{ is parallel.} \tag{4.4}$$

Indeed, the left side of (4.3) equals $-2(\nabla_z\Theta)(v_1,\ldots,v_k)$, due to (3.1) and the definition of $\nabla_z\Theta$. Bi-invariant magnetic systems will be discussed in more detail in Section 5, but we may already register a very useful characterization which follows from the above discussion.

PROPOSITION 4.1. *A left-invariant magnetic system* (g,σ) *on a connected Lie group* G *is bi-invariant if and only if, for every* $z\in\mathfrak{g}$, ad_z *is skew-adjoint and commutes with* $\mathbf{Y}$.

PROOF. We know from [**36**, Lemma 7.2] that g is bi-invariant if and only if for each $z\in\mathfrak{g}$, ad_z is skew-adjoint. In this case (3.1) yields $2\nabla_z\mathbf{Y}=[\mathrm{ad}_z,\mathbf{Y}]$ and, noting that σ is parallel if and only if $\mathbf{Y}$ is parallel, we may invoke (4.4). $\square$

We now proceed to compute the magnetic curvature of a left-invariant magnetic system (g,σ). The curvatures will always be evaluated at vectors in $\mathfrak{g}$, and so we omit the identity base point $e\in G$, using the respective shorthand notations $\mathbf{Y}$, $A_v^{(\mathrm{g},\sigma)}$, $\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v,w)$, and $\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(v)$ for $\mathbf{Y}_e$, $A_{(e,v)}^{(\mathrm{g},\sigma)}$, $\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(e,v)$, and

$\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(e,v)$, where $v,w\in\mathfrak{g}$. First, we claim that

$$\begin{aligned}\langle(\nabla_w\mathbf{Y})(v),w\rangle &= \langle[w,\mathbf{Y}(v)],w\rangle+\frac{1}{2}\langle[w,v],\mathbf{Y}(w)\rangle\\ &\quad-\frac{1}{2}\langle[v,\mathbf{Y}(w)],w\rangle-\frac{1}{2}\langle[w,\mathbf{Y}(w)],v\rangle\end{aligned}\tag{4.5}$$

holds, for any $v,w\in\mathfrak{g}$. Indeed, by definition of $\nabla_w\mathbf{Y}$ and skew-adjointness of $\mathbf{Y}$, we have the relation $\langle(\nabla_w\mathbf{Y})(v),w\rangle=\langle\nabla_w(\mathbf{Y}(v)),w\rangle+\langle\nabla_w v,\mathbf{Y}(w)\rangle$. Using (3.1) and that $[w,w]=0$, it follows that $\langle\nabla_w(\mathbf{Y}(v)),w\rangle=\langle[w,\mathbf{Y}(v)],w\rangle$. A similar calculation again using (3.1) shows that the second term $\langle\nabla_w v,\mathbf{Y}(w)\rangle$ amounts to the remaining three terms in (4.5). We register next the result of plugging (2.3), (3.3), and (4.5) into (2.7).

Proposition 4.2. *Let* (g,σ) *be any left-invariant magnetic system on* G*, and* $s>0$*. Then, we have that*

$$\begin{aligned}\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v,w) &= s^2\Big(-\frac{3}{4}\|[v,w]\|^2-\frac{1}{2}\langle[[v,w],w],v\rangle-\frac{1}{2}\langle[v,[v,w]],w\rangle\\ &\quad-\langle B(v,v),B(w,w)\rangle+\|B(v,w)\|^2\Big)-s\Big(\langle[w,\mathbf{Y}(v)],w\rangle\\ &\quad+\frac{1}{2}\langle[w,v],\mathbf{Y}(w)\rangle-\frac{1}{2}\langle[v,\mathbf{Y}(w)],w\rangle-\frac{1}{2}\langle[w,\mathbf{Y}(w)],v\rangle\Big)\\ &\quad+\frac{3}{4}\langle w,\mathbf{Y}(v)\rangle^2+\frac{1}{4}\|\mathbf{Y}(w)\|^2,\end{aligned}\tag{4.6}$$

whenever $v,w\in\mathfrak{g}$ *are orthonormal, for* B *as in* (3.1).

The next result generalizes [**36**, Lemma 1.6], which states that the Riemannian sectional curvature of a left-invariant metric is nonnegative along planes containing directions z with skew-adjoint ad_z. The new assumption considered below, which means that the one-parameter subgroup generated by z acts on G via conjugations preserving *both* g and σ, is of course motivated by Proposition 4.1.

Proposition 4.3. *Let* $z\in\mathfrak{g}$ *be a unit vector such that* ad_z *is skew-adjoint and commutes with* $\mathbf{Y}$*. Then,* $\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v,z)\geq 0$ *for every* $s>0$ *and unit vector* $v\in\mathfrak{g}$ *orthogonal to* z*, with equality if and only if* z *is orthogonal to the image of* $\mathrm{ad}_v-s^{-1}\mathbf{Y}$*. In particular, this holds if* z *is a central element.*

Proof. Using that z is orthogonal to the images of ad_z and $v\mapsto\mathrm{ad}_v^\dagger z$ whenever ad_z is skew-adjoint, a routine computation reduces (3.3) to $\mathrm{Sec}^{\mathrm{g}}(v,z)=\|\mathrm{ad}_v^\dagger z\|^2/4$. Next, using also the condition that $[\mathrm{ad}_z,\mathbf{Y}]=0$, only the third term in (4.5) remains, which we write as $\langle(\nabla_z\mathbf{Y})(v),z\rangle=-\langle\mathbf{Y}(z),\mathrm{ad}_v^\dagger z\rangle/2$. Substituting all of it into (2.7) and factoring it with (2.3), we arrive at

$$\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v,z)=\frac{s^2}{4}\|\mathrm{ad}_v^\dagger z+s^{-1}\mathbf{Y}(z)\|^2+\frac{3}{4}\langle\mathbf{Y}(z),v\rangle^2\geq 0,\tag{4.7}$$

with equality if and only if $\mathrm{ad}_v^\dagger z + s^{-1}\mathbf{Y}(z) = 0$ and $\mathbf{Y}(z)$ is orthogonal to v. However, the latter condition is implied by the former, and the conclusion follows from the general relation $\langle \mathrm{ad}_v^\dagger z + s^{-1}\mathbf{Y}(z), w\rangle = \langle z, \mathrm{ad}_v w - s^{-1}\mathbf{Y}(w)\rangle$, valid for all $w \in \mathfrak{g}$. □

Theorem A now follows from Proposition 4.3, in view of Proposition 4.1 together with the easily verified claim that $\mathrm{ad}_v^\dagger z + s^{-1}\mathbf{Y}(z)$ vanishes for all $v, z \in \mathfrak{g}$ if and only if $\mathrm{ad}_v = 0$ for all v (i.e., G is Abelian) and $\mathbf{Y} = 0$ (hence $\sigma = 0$). Alternatively, one may simplify (4.6) in the case where (g, σ) is bi-invariant to obtain

$$\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v,w) = \frac{1}{4}\left(s^2\|[v,w]\|^2 + 3\langle w, \mathbf{Y}(v)\rangle^2 + \|\mathbf{Y}(w)\|^2\right), \tag{4.8}$$

for all orthonormal $v, w \in \mathfrak{g}$.

Example 4.4. The assumption that ad_z commutes with $\mathbf{Y}$ in Proposition 4.3 cannot be dropped. Consider the Lie group $\mathrm{E}(2)$ of all rigid motions of the plane, equipped with the left-invariant metric g for which the basis x, y, z of $\mathfrak{e}(2)$ satisfying the bracket relations $[z,x] = -y$, $[z,y] = x$, and $[x,y] = 0$ declared to be g-orthonormal. Note that every left-invariant 2-form on $\mathrm{E}(2)$ is automatically closed: as $\mathrm{E}(2)$ is three-dimensional, it suffices to check that (4.1) is satisfied when evaluated at $(u,v,w) = (x,y,z)$; this is obvious from the given bracket relations. The operator $\mathbf{Y}$ defined by $\mathbf{Y}(x) = 0$, $\mathbf{Y}(y) = z$, $\mathbf{Y}(z) = -y$, is a rotation in the yz-plane, and thus skew-adjoint; define σ via (1.2). The operator ad_z is also skew-adjoint, being another rotation, but $[\mathrm{ad}_z, \mathbf{Y}](x) = z \neq 0$. Using (3.2), we compute that $\nabla_z x = -y$ and $\nabla_x z = \nabla_y z = \nabla_z z = 0$, so that $R^{\mathrm{g}}(x,z)z = 0$ and $\langle(\nabla_z \mathbf{Y})(x), z\rangle = 1$. From (2.7), we finally obtain that $\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(x,z) = -s + 1/4 < 0$ for all $s > 1/4$.

Example 4.5. In view of (2.5), one may also ask if the natural analogue of Proposition 4.3 holds, with $\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(z,w) \geq 0$ for all $s > 0$ and unit vector $w \in \mathfrak{g}$ orthogonal to z, provided that ad_z is skew-adjoint and commutes with $\mathbf{Y}$. This is, however, false: consider the group $\mathrm{Aff}(\mathbb{C})$ of affine functions on $\mathbb{C}$, equipped with the left-invariant metric g for which the basis x, y, z, w of $\mathfrak{aff}(\mathbb{C})$ having the nonzero bracket relations $[x,w] = w$, $[x,y] = y$, $[z,w] = y$, and $[z,y] = -w$, is g-orthonormal. Using (3.1) and (3.2), we fully describe the Levi-Civita connection of g in Table 1.

(4.9)

∇	x	y	z	w
x	0	0	0	0
y	$-y$	x	0	0
z	0	$-w$	0	y
w	$-w$	0	0	x

Table 1. The Levi-Civita product of g in $\mathfrak{aff}(\mathbb{C})$.

The operator $\mathbf{Y}$ defined by $\mathbf{Y}(x) = z$, $\mathbf{Y}(y) = 0$, $\mathbf{Y}(z) = -x$, and $\mathbf{Y}(w) = 0$, being a rotation in the xz-plane, is skew-adjoint and defines a closed 2-form σ, by (1.2) and (4.2). It also commutes with ad_z, which is a similar rotation in the yw-plane, and hence skew-adjoint as well. Using (2.7) and Table 1, we find that

$$\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(z,x) = 1, \quad \text{and} \quad \mathrm{Sec}^{(\mathrm{g},\sigma,s)}(z,y) = \mathrm{Sec}^{(\mathrm{g},\sigma,s)}(z,w) = -s, \tag{4.10}$$

for all $s > 0$, making the latter expression manifestly negative.

Remark 4.6. In [**36**, Lemmas 2.1 and 2.3], two results on Ricci curvature are presented: (i) If $z \in \mathfrak{g}$ has skew-adjoint ad_z, then $\mathrm{Ric}^{\mathrm{g}}(z) \geq 0$, with equality if and only if $z \in [\mathfrak{g},\mathfrak{g}]^\perp$, and (ii) if $z \in [\mathfrak{g},\mathfrak{g}]^\perp$, then $\mathrm{Ric}^{\mathrm{g}}(z) \leq 0$, with equality if and only if ad_z is skew-adjoint. Both are false in the magnetic setting. For (i), we recycle Example 4.5: it follows from (2.6) and (4.10) that $\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(z) = 1 - 2s < 0$ for $s > 1/2$. As for (ii), note from (2.3) that $\mathrm{tr}(A^{(\mathrm{g},\sigma)}_{(x,v)}) \geq 0$, with equality if and only if $\mathbf{Y}_x = 0$, so that making $s \to 0^+$ in (2.8) shows that $\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(x,v) > 0$ for s sufficiently small whenever $\mathbf{Y}_x \neq 0$, regardless of any properties one might require of v.

5. More on the bi-invariant case

It is well-known that a Lie group admits a bi-invariant metric if and only if G is isomorphic to the product of a compact group and an Abelian group [**36**, p. 297]. We may clearly replace "bi-invariant metric" with "bi-invariant magnetic system" here, by simply considering systems of the form $(\mathrm{g}, 0)$. This raises the question of when is it possible to find a *nontrivial* bi-invariant magnetic system, that is, one with $\sigma \neq 0$.

For example, rearranging (4.3), we have that

(5.1) a left-invariant bilinear form Θ is bi-invariant if and only if $\Theta([x,y],z) = \Theta(x,[y,z])$ for all $x,y,z \in \mathfrak{g}$.

With this, we claim that

$$\text{if } \mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \text{ (that is, } \mathfrak{g} \text{ is \textit{perfect}), the only bi-invariant bilinear form on } \mathfrak{g} \text{ is the trivial one.} \tag{5.2}$$

Indeed, if Θ is bi-invariant, the trilinear form $(x,y,z) \mapsto \Theta([x,y],z)$ is at the same time skew-symmetric in the first pair of arguments, and symmetric in the second pair by (5.1), and hence vanishes, implying that $\Theta([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = 0$, and thus $\Theta = 0$ as required. A similar argument shows that, in general, every bi-invariant 2-form is automatically closed: applying bi-invariance of σ to each term in the left side of (4.1) yields $\mathrm{d}\sigma = -\mathrm{d}\sigma$.

Below, recall that a Lie algebra $\mathfrak{g}$ is called *reductive* if there is a direct-sum decomposition

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \bigoplus_{i=1}^{k} \mathfrak{g}_i, \quad \text{with each of } \mathfrak{g}_1,\dots,\mathfrak{g}_k \text{ simple,} \tag{5.3}$$

where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. This is a natural condition to consider, being automatically satisfied whenever G admits a bi-invariant metric. Namely, in this case (5.1) ensures that $\mathfrak{z}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]^\perp$, yielding

$$\text{the orthogonal direct-sum splitting } \mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}], \tag{5.4}$$

with $[\mathfrak{g},\mathfrak{g}]$ semisimple: by (5.4), any Abelian ideal $\mathfrak{a} \subseteq [\mathfrak{g},\mathfrak{g}]$ must in fact commute with all of $\mathfrak{g}$, leading to $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}] = \{0\}$.

Proposition 5.1. *If $\mathfrak{g}$ is reductive, the restriction mapping*

$$\left(\wedge^2\mathfrak{g}^*\right)^G \ni \sigma \mapsto \sigma|_{\mathfrak{z}(\mathfrak{g})\times\mathfrak{z}(\mathfrak{g})} \in \wedge^2\mathfrak{z}(\mathfrak{g})^* \tag{5.5}$$

is an isomorphism; here, $\left(\wedge^2\mathfrak{g}^*\right)^G$ denotes the space of bi-invariant 2-forms on $\mathfrak{g}$.

Proof. For a decomposition of $\mathfrak{g}$ as in (5.3), and any bi-invariant 2-form σ on G, we first claim that

$$\text{(i) } \sigma(\mathfrak{g}_i,\mathfrak{g}_j) = 0, \text{ and (ii) } \sigma(\mathfrak{g}_i,\mathfrak{z}(\mathfrak{g})) = 0, \text{ for all } i,j = 1,\dots,k. \tag{5.6}$$

If $i = j$, (5.6-i) follows from (5.2) as each $\mathfrak{g}_i$, being simple, is also perfect. Assuming that $i \neq j$ instead, let $\mathfrak{b}$ be either $\mathfrak{g}_j$ or $\mathfrak{z}(\mathfrak{g})$, take $x,y \in \mathfrak{g}_i$ and $z \in \mathfrak{b}$, and note that $\sigma([x,y],z) = \sigma(x,[y,z]) = 0$ as $[\mathfrak{g}_i,\mathfrak{b}] = 0$, so that $\sigma([\mathfrak{g}_i,\mathfrak{g}_i],\mathfrak{b}) = 0$, leading to both (5.6-i) and (5.6-ii) due to perfectness of $\mathfrak{g}_i$. Finally, condition (5.6) amounts to injectivity of (5.5), while its surjectivity is clear: given a 2-form on $\mathfrak{z}(\mathfrak{g})$, extend it by zero via (5.3) and (5.6) to a 2-form on $\mathfrak{g}$; bi-invariance of the resulting extension is immediate from (5.1), as all relevant brackets vanish. $\square$

We may now establish the main result of this section.

THEOREM 5.2. *A connected Lie group G admits a* non-trivial *bi-invariant magnetic system if and only if G is isomorphic to a direct product $K \times \mathbb{R}^m$, where $\mathbb{R}^m$ is Abelian and K is a compact Lie group with* $\dim \mathfrak{z}(\mathfrak{k}) + m \geq 2$.

PROOF. It follows from Proposition 5.1 that, in the presence of a bi-invariant metric, nonzero bi-invariant 2-forms exist if and only if $\dim \mathfrak{z}(\mathfrak{g}) \geq 2$; this is precisely the condition needed for $\dim \left(\wedge^2 \mathfrak{g}^*\right)^G > 0$. We now invoke [**36**, p. 297], stated in the beginning of this section, and note that $\dim \mathfrak{z}(\mathfrak{g}) = \dim \mathfrak{z}(\mathfrak{k}) + m$. □

COROLLARY 5.3. *A connected and semisimple Lie group does not admit non-trivial bi-invariant magnetic systems.*

EXAMPLE 5.4. If g is a bi-invariant metric on G, we may consider $\mathbf{Y} = \mathrm{ad}_\xi$ for some $\xi \in \mathfrak{g}$; the corresponding 2-form σ_ξ obtained via (1.2) is closed: (4.1) becomes the Jacobi identity. This example, at first natural, has a crucial "flaw": if (g, σ_ξ) is bi-invariant, then $\sigma_\xi = 0$. Indeed, in this case Proposition 4.1 tells us that $0 = [\mathrm{ad}_\xi, \mathrm{ad}_x] = \mathrm{ad}_{[\xi,x]}$ for all $x \in \mathfrak{g}$, so that $[\xi, x] \in \mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$, cf. (5.4), and thus $\mathbf{Y} = \mathrm{ad}_\xi = 0$. When g is bi-invariant and σ_ξ is not, from (2.7) and (2.6) we still obtain

$$
\begin{aligned}
\mathrm{Sec}^{(\mathrm{g},\sigma_\xi,s)}(v,w) &= \frac{1}{4}\|[sv-\xi,w]\|^2 + \frac{3}{4}\langle[\xi,v],w\rangle^2 \\
\text{and}\quad \mathrm{Ric}^{(\mathrm{g},\sigma_\xi,s)}(v) &= -\frac{1}{4}\beta(sv-\xi,sv-\xi) + \frac{3}{4}\|[\xi,v]\|^2
\end{aligned} \tag{5.7}
$$

for all orthonormal $v, w \in \mathfrak{g}$, where β is the Killing form of $\mathfrak{g}$.

6. Magnetic flatness of left-invariant systems

The meaning of magnetic flatness for a general magnetic system (g, σ) on a smooth manifold M is well-understood:

(6.1) If $s > 0$ is such that $\mathrm{Sec}^{(\mathrm{g},\sigma,s)} = 0$, then $\nabla\sigma = 0$ and one of the following must hold: (i) $\sigma = 0$ and g is flat, or (ii) σ is nowhere-vanishing and $J = \|\mathbf{Y}\|^{-1}\mathbf{Y}$ turns (M, g) into a Kähler manifold of constant negative holomorphic sectional curvature $-\|\mathbf{Y}\|^2/s^2$,

cf. [**9**, Theorem D]. Here, $\|\mathbf{Y}\|$ denotes the operator norm of $\mathbf{Y}$. The goal of this section is to refine (6.1) in the case of left-invariant magnetic systems on Lie groups, by proving Theorems B and C. One general observation we will need is that if a Kähler manifold has constant holomorphic sectional curvature k, then its sectional curvature is bounded between $k/4$ and k, cf. [**30**, Propositions 7.3 and 7.4]; in particular, it follows that

(6.2) if $k < 0$, both the sectional and Ricci curvatures are also strictly negative.

In both Theorems B and C the situation where $\sigma = 0$ amounts to [**36**, Theorems 1.5 and 1.6], and so it suffices to address the case where σ is nowhere-vanishing.

Proof of Theorem B. If σ is nowhere-vanishing, it follows from (6.1-ii) and (6.2) that g is negatively curved, allowing us to invoke the result by Heintze mentioned in the Introduction—it follows that (G, g) with the complex structure $J = \|\mathbf{Y}\|^{-1}\mathbf{Y}$ is holomorphically isometric to a $\mathbb{CH}^n$ [**27**, Theorem 4]. Conversely, if (G, g) equipped with $J = \|\mathbf{Y}\|^{-1}\mathbf{Y}$ is holomorphically isometric to a $\mathbb{CH}^n$ with constant holomorphic curvature $k < 0$, then (g, σ) is in particular a Kähler magnetic system (as $\|\mathbf{Y}\|$ is constant by left-invariance of g and $\mathbf{Y}$), with s-magnetic sectional curvature given by

$$\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v, w) = \frac{s^2k + \|\mathbf{Y}\|^2}{4}(1 + 3\langle v, Jw\rangle^2), \tag{6.3}$$

cf. [**9**, Formula (6.2)]; it vanishes identically for $s = \|\mathbf{Y}\|/\sqrt{-k}$. □

Proof of Theorem C. As in the previous proof, the sectional curvature is strictly negative. By [**27**, Remark in p. 27], the unique connected and simply connected Lie group $\widetilde{G}$ having $\mathfrak{g}$ as its Lie algebra is solvable. Being connected, G is a quotient of $\widetilde{G}$, and hence solvable as well. Finally, if $\sigma \neq 0$, G cannot be unimodular, as otherwise (6.2) contradicts a classical result by Dotti: no solvable unimodular Lie group admits a left-invariant Riemannian metric of strictly negative Ricci curvature [**22**, Corollary 3.3]. □

7. The Heisenberg group

Let (V, Ω) be a symplectic vector space, and denote by $\mathrm{H}(V, \Omega)$ its abstract Heisenberg group. Namely, $\mathrm{H}(V, \Omega)$ is the Cartesian product $V \times \mathbb{R}$ equipped with the group operation $(x, a)(y, b) = (x + y, a + b + \Omega(x, y)/2)$, and its Lie algebra $\mathfrak{h}(V, \Omega) = V \times \mathbb{R}$ has the bracket $[(v, a), (w, b)] = (0, \Omega(v, w))$. The group $\mathrm{H}(V, \Omega)$ is two-step nilpotent, its center equals $\{0\} \times \mathbb{R}$ due to nondegeneracy of Ω, and so the resulting short exact sequence $0 \to \mathbb{R} \to \mathrm{H}(V, \Omega) \to V \to 0$ yields that $\mathrm{H}(V, \Omega)$ is a central extension of V by $\mathbb{R}$. Let $\langle \cdot, \cdot \rangle_0$ be any inner product on V, and consider the left-invariant magnetic system (g, σ) on $\mathrm{H}(V, \Omega)$ given by

$$\mathrm{g}((v, a), (w, b)) = \langle v, w\rangle_0 + ab \quad \text{and} \quad \sigma((v, a), (w, b)) = \Omega(v, w) \tag{7.1}$$

on $\mathfrak{h}(V, \Omega)$. From (4.2), σ is closed; in fact, $\sigma = -\mathrm{d}\alpha$, where $\alpha\colon \mathfrak{h}(V, \Omega) \to \mathbb{R}$ denotes the second coordinate projection. Note that all left-invariant magnetic systems on $\mathrm{H}(V, \Omega)$ having a left-invariant potential 1-form and making V and $\mathbb{R}$ orthogonal are, up to rescaling, of the form (7.1). The Lorentz force operator of

(g, σ) is given by $\mathbf{Y}(v, a) = (\mathbf{Y}_0(v), 0)$, where $\mathbf{Y}_0$ is the operator on V characterized $\langle \mathbf{Y}_0(v), w\rangle_0 = \Omega(v, w)$ for all $v, w \in V$. We similarly have that

$$\mathrm{ad}_{(v,a)}(w, b) = (0, \Omega(v, w)) \quad \text{and} \quad \mathrm{ad}^\dagger_{(v,a)}(w, b) = (b\mathbf{Y}_0(v), 0), \tag{7.2}$$

and hence the only elements (v, a) for which $\mathrm{ad}_{(v,a)}$ is skew-adjoint are precisely the ones with $v = 0$. Setting $\mathbf{z} = (0, 1)$ and identifying (v, a) with $v + a\mathbf{z}$, we use (3.1), (3.2), and (7.2) to obtain

$$\nabla_v w = \frac{1}{2}\Omega(v, w)\mathbf{z}, \quad \nabla_{\mathbf{z}} v = \nabla_v \mathbf{z} = -\frac{1}{2}\mathbf{Y}_0(v), \quad \text{and} \quad \nabla_{\mathbf{z}}\mathbf{z} = 0. \tag{7.3}$$

Noting that $\mathbf{z}$ is central with $\mathbf{Y}(\mathbf{z}) = 0$, and $\mathrm{g}([u, v], w) = 0$ for all $u, v, w \in V$, we use (7.3) to evaluate the three coefficients in (2.7), so that

$$\begin{aligned} \mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v + a\mathbf{z}, w + b\mathbf{z}) = s^2 \bigg(&\frac{a^2}{4}\|\mathbf{Y}_0(w)\|^2 + \frac{b^2}{4}\|\mathbf{Y}_0(v)\|^2 \\ &- \frac{3}{4}\Omega(v, w)^2 - \frac{ab}{2}\langle \mathbf{Y}_0(v), \mathbf{Y}_0(w)\rangle \bigg) \\ &+ s \left(\frac{a}{2}\|\mathbf{Y}_0(w)\|^2 - \frac{b}{2}\langle \mathbf{Y}_0(v), \mathbf{Y}_0(w)\rangle \right) \\ &+ \frac{3}{4}\Omega(v, w)^2 + \frac{1}{4}\|\mathbf{Y}_0(w)\|^2 \end{aligned} \tag{7.4}$$

for all orthonormal $v + a\mathbf{z}, w + b\mathbf{z} \in \mathfrak{h}(V, \Omega)$. Notable particular instances of (7.4), for orthonormal $v, w \in V$, are

$$\begin{aligned} &\text{(i) } \mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v, w) = \frac{1}{4}\left(3(1 - s^2)\Omega(v, w)^2 + \|\mathbf{Y}_0(w)\|^2\right), \\ &\text{(ii) } \mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v, \pm\mathbf{z}) = \frac{s^2}{4}\|\mathbf{Y}_0(v)\|^2, \quad \text{and} \\ &\text{(iii) } \mathrm{Sec}^{(\mathrm{g},\sigma,s)}(\pm\mathbf{z}, w) = \frac{(s \pm 1)^2}{4}\|\mathbf{Y}_0(w)\|^2. \end{aligned} \tag{7.5}$$

We compute the s-magnetic Ricci curvature of (g, σ) as well, using relation (2.6). Setting $\|\mathbf{Y}_0\| = \sqrt{-\mathrm{tr}(\mathbf{Y}_0^2)} > 0$, it is clear from (7.5-iii) that

$$\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(\pm\mathbf{z}) = \frac{(s \pm 1)^2}{4}\|\mathbf{Y}_0\|^2, \tag{7.6}$$

and it remains to find $\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(v + a\mathbf{z})$, for a unit vector $v + a\mathbf{z} \in \mathfrak{h}(V, \Omega)$ with $v \neq 0$ (which itself does *not* have unit length). One orthonormal basis for the orthogonal complement of $v + a\mathbf{z}$ in $\mathfrak{h}(V, \Omega)$ is $e_1, \ldots, e_{n-1}, -a\|v\|_0^{-1}v + \|v\|_0\mathbf{z}$, where $e_1, \ldots, e_{n-1}$ is an orthonormal basis for the orthogonal complement of v in V, cf. Figure 1.

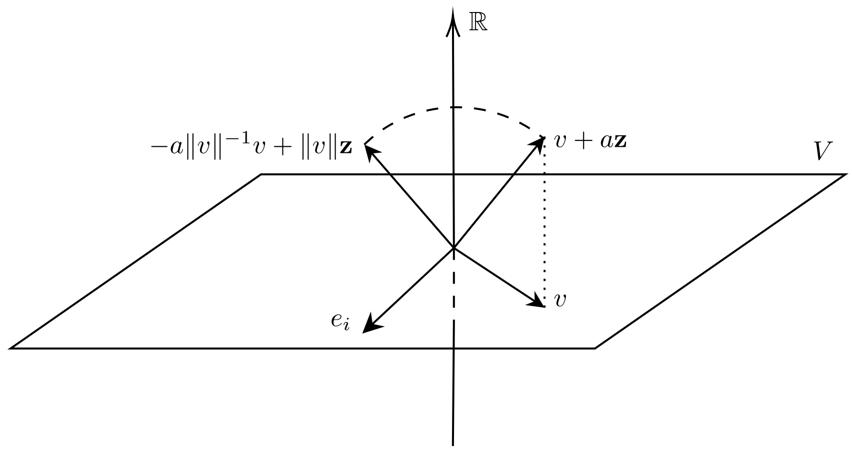

FIGURE 1. An orthonormal basis for $(v + a\mathbf{z})^\perp \subseteq \mathfrak{h}(V, \Omega)$.

The unit-length condition $\|v\|^2 + a^2 = 1$ allows for significant simplifications in the expressions to come. Firstly, (7.4) yields

$$\mathrm{Sec}^{(\mathrm{g},\sigma,s)}\left(v + a\mathbf{z}\,,\, -a\|v\|^{-1}v + \|v\|\mathbf{z}\right) = \frac{(s+a)^2}{4}\frac{\|\mathbf{Y}_0(v)\|^2}{\|v\|^2}, \tag{7.7}$$

while

$$\sum_{i=1}^{n-1}\|\mathbf{Y}_0(e_i)\|^2 = \|\mathbf{Y}_0\|^2 - \frac{\|\mathbf{Y}_0(v)\|^2}{\|v\|^2} \quad \text{and} \quad \sum_{i=1}^{n-1}\Omega(v, e_i)^2 = \|\mathbf{Y}_0(v)\|^2 \tag{7.8}$$

now imply that

$$\begin{aligned}\sum_{i=1}^{n-1}\mathrm{Sec}^{(\mathrm{g},\sigma,s)}(v + a\mathbf{z}, e_i) = \frac{(1+sa)^2}{4}&\left(\|\mathbf{Y}_0\|^2 - \frac{\|\mathbf{Y}_0(v)\|^2}{\|v\|^2}\right)\\ &+ \frac{3(1-s^2)}{4}\|\mathbf{Y}_0(v)\|^2.\end{aligned} \tag{7.9}$$

Adding (7.7) to (7.9) and factoring further leads to the surprisingly concise formula

$$\mathrm{Ric}^{(\mathrm{g},\sigma,s)}(v + a\mathbf{z}) = \frac{(1+sa)^2}{4}\|\mathbf{Y}_0\|^2 + \frac{(1-s^2)}{2}\|\mathbf{Y}_0(v)\|^2. \tag{7.10}$$

It follows from (7.6) and (7.10) that $\mathrm{Ric}^{(\mathrm{g},\sigma,s)} > 0$ whenever $0 < s < 1$, while $\mathrm{Ric}^{(\mathrm{g},\sigma,s_0)}(-\mathbf{z}) = 0$ for $s_0 = 1$ by (7.6), correctly hinting that the Mañé critical value—which marks the most drastic changes in the geometric and dynamical behavior of the corresponding magnetic flow—is $\mathbf{c}(\mathrm{g}, \sigma) = s_0^2/2 = 1/2$, cf. [**2**, Theorem 2.4] and [**25**, Theorem 1].

Appendix A. A magnetic Euler-Arnold equation for exact systems

It is well-known [**3**,**4**] that if G is a Lie group and g is a left-invariant Riemannian metric on G, then a curve $\gamma\colon I \to G$ is a geodesic if and only if $x(t) = \gamma(t)^{-1}\dot{\gamma}(t)$ satisfies the so-called *Euler-Arnold equation* $\dot{x}(t) = \mathrm{ad}^{\dagger}_{x(t)}x(t)$. For completeness of our presentation, we exhibit a magnetic generalization of this formalism in Proposition A.2; it has already been considered, for instance, in [**33**, **38**].

We follow the strategy adopted in [**18**, Proposition 2.1]. The first ingredient needed is what they call the "zero-curvature" equation, and a reference likely to [**36**, Formula (5.1)]. We instead provide a direct proof.

Lemma A.1 (The zero-curvature equation). *Given a smooth curve $\gamma\colon I \to G$ and a variation $\eta\colon I \times (-\varepsilon, \varepsilon) \to G$ of γ, the translated partial derivatives*

$$u(t,s) = \eta(t,s)^{-1}\partial_t\eta(t,s) \quad \text{and} \quad v(t,s) = \eta(t,s)^{-1}\partial_s\eta(t,s) \tag{A.1}$$

satisfy the equation $\partial_s u - \partial_t v = \mathrm{ad}_u v$.

Proof. Consider the Maurer-Cartan form $\Theta \in \Omega^1(G, \mathfrak{g})$ of G, defined by $\Theta_g(v) = g^{-1}v$, and its well-known structure equation

$$\mathrm{d}\Theta(X,Y) + [\Theta(X), \Theta(Y)] = 0, \quad \text{for all } X, Y \in \mathfrak{X}(G), \tag{A.2}$$

cf. [**28**, Proposition 7.3], which corresponds to flatness of Θ as a connection 1-form on the principal G-bundle $G \to \{e\}$. Fix also any torsionfree connection ∇ on G, so that the formula

$$\mathrm{d}\Theta(X,Y) = (\nabla_X\Theta)(Y) - (\nabla_Y\Theta)(X) \tag{A.3}$$

holds. Finally, noting that $u = \Theta(\partial_t\eta)$ and $v = \Theta(\partial_s\eta)$, we compute:

$$\begin{aligned} \partial_s u - \partial_t v &= \partial_s(\Theta(\partial_t\eta)) - \partial_t(\Theta(\partial_s\eta)) \\ &= (\nabla_{\partial_s\eta}\Theta)(\partial_t\eta) + \Theta(\nabla_{\partial_s\eta}\partial_t\eta) - (\nabla_{\partial_t\eta}\Theta)(\partial_s\eta) - \Theta(\nabla_{\partial_t\eta}\partial_s\eta) \\ &= \mathrm{d}\Theta(\partial_s\eta, \partial_t\eta) + \Theta(\nabla_{\partial_s\eta}\partial_t\eta - \nabla_{\partial_t\eta}\partial_s\eta) \\ &= -[\Theta(\partial_s\eta), \Theta(\partial_t\eta)] + 0 \\ &= -[v,u] \\ &= \mathrm{ad}_u v, \end{aligned} \tag{A.4}$$

as required; in the third and fourth lines of (A.4) we use (A.2) and (A.3), respectively, as well as the equality $\nabla_{\partial_s\eta}\partial_t\eta = \nabla_{\partial_t\eta}\partial_s\eta$, valid as ∇ is torsionfree. $\square$

The second ingredient is a variational characterization of magnetic geodesics for exact magnetic systems. Namely, if $(\mathrm{g}, -\mathrm{d}\alpha)$ is an exact magnetic system on a smooth manifold M, then a smooth curve $\gamma\colon [a,b] \to M$ is a $(\mathrm{g}, -\mathrm{d}\alpha)$-geodesic if and only if it is a critical point (under variations with fixed endpoints) of the *Landau-Hall functional*

$$E(\gamma) = \int_a^b \left(\frac{1}{2}\|\dot\gamma(t)\|^2 + \alpha_{\gamma(t)}(\dot\gamma(t))\right) \mathrm{d}t, \tag{A.5}$$

cf. [**13**, Section I] (specifically, the Euler-Lagrange equation of (A.5) is precisely (1.1)).

Below, suppose that $(\mathrm{g}, -\mathrm{d}\alpha)$ is a left-invariant magnetic system on the Lie group G, with the potential 1-form α also left-invariant.

Proposition A.2. *A smooth curve $\gamma\colon [a,b] \to G$ is a $(\mathrm{g}, -\mathrm{d}\alpha)$-geodesic if and only if $x(t) = \gamma(t)^{-1}\dot\gamma(t)$ satisfies the* magnetic Euler-Arnold equation

$$\dot x(t) = \mathrm{ad}^\dagger_{x(t)}(x(t) + \alpha^\sharp), \tag{A.6}$$

where $\alpha^\sharp \in \mathfrak{g}$ is the vector g*-dual to α. In particular, $(\mathrm{g}, -\mathrm{d}\alpha)$-geodesics are in one-to-one correspondence with integral curves of the* magnetic Euler-Arnold vector field *$Z \in \mathfrak{X}(\mathfrak{g})$, defined by $Z_x = \mathrm{ad}^\dagger_x(x + \alpha^\sharp)$ for all $x \in \mathfrak{g}$.*

Proof. Let η be a variation of γ with fixed endpoints, and consider the translated partial derivatives u and v of η as in (A.1). By left-invariance of g and α, we may rewrite (A.5) as

$$E(\eta(\cdot, s)) = \int_a^b \left(\frac{1}{2}\|u(t,s)\|^2 + \alpha_e(u(t,s))\right) \mathrm{d}t. \tag{A.7}$$

Differentiating (A.7) with respect to s (omitting arguments and setting $\mathrm{g} = \langle\cdot,\cdot\rangle$), we obtain

$$\begin{aligned}
\partial_s E(\eta) &= \int_a^b \langle \partial_s u, u\rangle + \alpha(\partial_s u)\,\mathrm{d}t \\
&\overset{(*)}{=} \int_a^b \langle \mathrm{ad}_u v + \partial_t v, u\rangle + \alpha(\mathrm{ad}_u v + \partial_t v)\,\mathrm{d}t \\
&\overset{(**)}{=} \int_a^b \langle v, \mathrm{ad}^\dagger_u u - \partial_t u\rangle + \langle \alpha^\sharp, \mathrm{ad}_u v\rangle\,\mathrm{d}t \\
&= \int_a^b \langle v, \mathrm{ad}^\dagger_u u - \partial_t u + \mathrm{ad}^\dagger_u(\alpha^\sharp)\rangle\,\mathrm{d}t.
\end{aligned} \tag{A.8}$$

In $(*)$ we have used the zero-curvature equation, and in $(**)$ integration by parts (noting that all boundary terms vanish as η has fixed endpoints) together with $\alpha = \langle\alpha^\sharp, \cdot\rangle$. Thus γ is a critical point of E if and only if $\mathrm{ad}^\dagger_x x - \dot x + \mathrm{ad}^\dagger_x(\alpha^\sharp) = 0$, which is evidently equivalent to (A.6). $\square$

Remark A.3. The magnetic Euler-Arnold vector field does not depend on the choice of α, in the following sense: if β is another left-invariant 1-form such that

$\mathrm{d}\alpha = \mathrm{d}\beta$, then $\mathrm{ad}_x^\dagger(x + \alpha^\sharp) = \mathrm{ad}_x^\dagger(x + \beta^\sharp)$ for all $x \in \mathfrak{g}$. Indeed, $\mathrm{d}(\alpha - \beta) = 0$ means that $(\alpha - \beta)([x, y]) = 0$, and so $\langle \mathrm{ad}_x^\dagger(\alpha^\sharp - \beta^\sharp), y\rangle = \langle \alpha^\sharp - \beta^\sharp, [x, y]\rangle = 0$ for all $x, y \in \mathfrak{g}$. Hence $\mathrm{ad}_x^\dagger(\alpha^\sharp - \beta^\sharp) = 0$, as claimed.

Remark A.4. Every left-invariant magnetic system is complete. Indeed, (g, σ) is complete whenever g is complete [**13**, Corollary 2.4] and, more generally, every homogeneous Riemannian metric is complete [**31**, Corollary 4.4].

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